Prove that if P is an arbitrary population and W is the set of z-scores generated from P, then W has mean 0 and standard deviation 1.

Proof: Let μ_P and σ_P represent the mean and standard deviation of P, and let μ_Z and σ_Z represent the mean and standard deviation of W. Let N be the population size, and note that N is also the number of z-scores, since there is one z-score generated for each member of the population. Then by definition of the mean,

$$\mu_Z = \frac{\Sigma_{z \epsilon W}\left(z\right)}{N}$$

Using the definition of the z-score, each z-score satisfies $z = \frac{x - \mu_P}{\sigma_P}$ for some x in the population. This gives

$$\mu_Z = \frac{\Sigma_{x \epsilon P}\left(\frac{x-\mu_P}{\sigma_P}\right)}{N}$$

We can factor out the $\frac{1}{\sigma_P}$ and break apart the summation to get

$$\mu_Z = \frac{1}{\sigma_P N}\left(\Sigma_{x \epsilon P}\left(x\right) - \Sigma_{x \epsilon P}\left(\mu_P\right)\right)$$

Using the definition of μ_P gives $\mu_P = \frac{\Sigma_{x \epsilon P}(x)}{N}$, and multiplying both sides of this last equation by N gives $N\mu_P = \Sigma_{x \epsilon P}\left(x\right)$. But we also have $\Sigma_{x \epsilon P}\left(\mu_P\right) = N\mu_P$, since we are adding up N copies of μ_P (one for each member of the population). This gives

$$\mu_Z = \frac{1}{N\sigma_P}\left(N\mu_P - N\mu_P\right) = 0$$

Using the definition of the standard deviation of the z-scores,

$$\sigma_Z = \sqrt{\frac{\Sigma_{z \epsilon W}\left(\left(z - \mu_Z\right)^2\right)}{N}}$$

Using the fact that $\mu_Z = 0$ gives

$$\sigma_Z = \sqrt{\frac{\Sigma_{z \epsilon W}\left(z^2\right)}{N}}$$

Using the definition of the z-score again gives

$$\sigma_Z = \sqrt{\frac{\Sigma_{x \epsilon P}\left(\left(\frac{x-\mu_P}{\sigma_P}\right)^2\right)}{N}}$$

We can factor out the $\frac{1}{\sigma_P^2}$ from the summation to get

$$\sigma_Z = \sqrt{\frac{\Sigma_{x \epsilon P}\left(\left(x - \mu_P\right)^2\right)}{\sigma_P^2 N}}$$

We can pull the $\frac{1}{\sigma_P^2}$ outside the square root to get

$$\sigma_Z = \frac{1}{\sigma_P}\sqrt{\frac{\Sigma_{x \epsilon P}\left(\left(x - \mu_P\right)^2\right)}{N}}$$

But by definition, $\sigma_P = \sqrt{\frac{\Sigma_{x \epsilon P}\left(\left(x-\mu_P\right)^2\right)}{N}}$, so

$$\sigma_Z = \frac{1}{\sigma_P}\sigma_P = 1$$