3. SECTIONAL CURVATURE OF LORENTZIAN MANIFOLDS.

1. Sectional curvature, the Jacobi equation and "tidal stresses".

The (3,1) Riemann curvature tensor has the same definition in the riemannian and Lorentzian cases:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

If $f(t,s)$ is a parametrized 2-surface in M (immersion) and $W(t,s)$ is a vector field on M along f, we have the *Ricci formula*:

$$\frac{D}{\partial t}\frac{DW}{\partial s} - \frac{D}{\partial s}\frac{DW}{\partial t} = R(\partial_t f, \partial_s f)W.$$

For a variation $f(t,s) = \gamma_s(t)$ of a geodesic $\gamma(t)$ (with variational vector field $J(t) = \partial_s f_{|s=0}$ along $\gamma(t)$) this leads to the *Jacobi equation* for J:

$$\frac{D^2 J}{dt^2} + R(J, \dot\gamma)\dot\gamma = 0.$$

The *Jacobi operator* is the self-adjoint operator on $(\dot\gamma)^\perp$: $\mathcal{R}_p[v] = R_p(v, \dot\gamma)\dot\gamma$.

When γ is a timelike geodesic (the worldline of a free-falling massive particle) the physical interpretation of J is the relative displacement (spacelike) vector of a neighboring free-falling particle, while the second covariant derivative J'' represents its relative acceleration. The Jacobi operator $\mathcal{R}_p$ gives the "tidal stresses" in terms of the position vector J.

In the Lorentzian case, the sectional curvature is defined only for *non-degenerate* two-planes $\Pi \subset T_p M$.

Definition. Let $\Pi = span\{X, Y\}$ be a non-degenerate two-dimensional subspace of $T_p M$. The sectional curvature $\sigma_{XY} = \sigma_\Pi$ is the real number σ defined by:

$$\langle R(X,Y)Y, X\rangle = \sigma \langle X \wedge Y, X \wedge Y\rangle.$$

Remark: by a result of J. Thorpe, σ does not extend continuously to degenerate two-planes, unles it is constant on non-degenerate ones.

Remark/exercise. Let $(V, \langle .,. \rangle)$ be a Lorentzian vector space of dimension $n+1$. The usual inner product:

$$\langle X \wedge Y, Z \wedge W\rangle = \langle X, Z\rangle\langle Y, W\rangle - \langle Y, Z\rangle\langle X, W\rangle,$$

in space $\Lambda_2(V)$ of alternating bivectors (of dimension $n(n+1)/2$) is non-degenerate, with signature $(n, n(n-1)/2)$ (index n). If $\Pi = span\{X, Y\}$, we have $\langle X \wedge Y, X \wedge Y\rangle$ negative iff Π is Lorentzian, zero iff Π is degenerate, positive iff Π is spacelike. The number σ depends only on Π, not on the chosen basis $\{X, Y\}$.

Exercise 1: Prove the assertions in this remark.

Interpreting the sign of σ. In the Riemannian case, if $\sigma < 0$ everywhere, the Jacobi equation has exponentially divergent solutions, while if $\sigma > 0$ the solutions are oscillatory ("convergent" geodesics.) This still holds for spacelike two-planes in the Lorentzian case: $\sigma < 0$ corresponds to defocusing, $\sigma > 0$ to focusing behavior.

If γ is a timelike geodesic, $J(t)$ a perpendicular Jacobi field (hence J and J'' are spacelike), the 2-planes $\Pi(t) = span\{\gamma', J(t)\}$ are Lorentzian, and $\langle J \wedge \gamma', J \wedge \gamma'\rangle = -|J|^2$ if $|\gamma'| = 1$. We have for the "radial component of the relative acceleration":

$$\langle J'', \frac{J}{|J|}\rangle = -\frac{\langle R(J,\gamma')\gamma', J\rangle}{|J|} = \sigma|J|.$$

This is positive if $\sigma > 0$ (divergent, defocusing behavior), negative if $\sigma < 0$ (convergent, focusing behavior for nearby particles.)

Note this is exactly the opposite of the Riemannian case.

2. Sectional curvature of Lorentzian hyperquadrics.

First we write down the Gauss and Codazzi equations for a non-degenerate hypersurface M (Riemannian or Lorentzian) in a Lorentzian manifold $\bar{M}$. Denoting by $II(X, Y)$ the vector-valued second fundamental form (with values in the normal bundle of M), we have the tangent-normal decompoistion of $\bar{\nabla}_X Y$ $(X \in \chi_M, Y \in \bar{\chi}_M)$:

$$\bar{\nabla}_X Y = \nabla_X Y + II(X,Y) = \epsilon_N A(X,Y)N, \quad A(X,Y) = \langle II(X,Y), N\rangle = \langle \bar{\nabla}_X Y, N\rangle = -\langle \bar{\nabla}_X N, Y\rangle,$$

where N is a choice of unit normal and $\epsilon_N = \langle N, N\rangle$.

The same derivation as in the Riemannian case gives for the (4,0) Riemann curvature tensors:

$$\bar{Riem} = Riem + \epsilon_N A \odot A,$$

and for the sectional curvatures along a non-degenerate 2-plane $\Pi = span\{X, Y\} \subset T_u M$ (orthonormal):

$$\bar{\sigma}_{XY} = \sigma_{XY} - \epsilon_N \epsilon_\Pi [A(X,X)A(Y,Y) - A(X,Y)^2],$$

where ϵ_Π equals 1 if Π is spacelike, -1 if Π is Lorentzian. If M is totally umbilic in $\bar{M}$ with normal curvature vector z (so $II(X,Y) = \langle X, Y\rangle z$), this simplifies to:

$$\bar{\sigma}_{XY} = \sigma_{XY} - \epsilon_N \langle z, N\rangle^2,$$

where N is a choice of unit normal.

Somewhat surprisingly, the Codazzi equation is *unchanged*:

$$\langle \bar{R}(X,Y)Z, N\rangle = (\nabla_X A)(Y,Z) - (\nabla_Y A)(X,Z).$$

In terms of the shape operator defined by $\langle S(X), Y\rangle = A(X,Y)$:

$$(\nabla_X S)Y - (\nabla_Y S)X = -\bar{R}(X,Y)N.$$

This simplifies if $\bar{M}$ has constant sectional curvature, for then $\bar{R}(X,Y)N = 0$ if $X, Y \in \chi_M$.

Exercise 2: Verify the claims just made about the Codazzi equation, including the remark regarding constant curvature spaces.

The standard hyperquadrics in Minkowski space $\mathbb{M}^{n+1}$ (which is flat) have constant sectional curvature:

Hyperbolic space $H^n(r) = \{u | \langle u, u\rangle = -r^2, u_0 > 0\}$. The normal curvature vector is $z(u) = \frac{1}{r}u$ (timelike) so $\sigma_{XY} = \epsilon_N \langle z, N\rangle^2 = -\frac{1}{r^2}$.

deSitter spacetime $S^n_1(r) = \{u | \langle u, u\rangle = r^2\}$. The normal curvature vector is $z(u) = -\frac{1}{r}u$ (spacelike), so $\sigma_{XY} = \epsilon_N \langle z, N\rangle^2 = \frac{1}{r^2}$.

Note this implies timelike geodesics in deSitter space exhibit *defocusing*, spacelike geodesics *focusing* behavior. The Ricci curvature of a timelike "unit" vector U is $Ric(U,U) = -(n-1)/r^2$.

3. Totally umbilic and constant curvature hypersurfaces.

Proposition 1. Let $M \subset \bar{M}$ be a hypersurface in a Lorentzian manifold of constant (sectional curvature $\bar{C}$. If M is totally umbilic in $\bar{M}$, then M has constant curvature.

Proof. Writing $II(X,Y) = \langle X, Y\rangle z$ and $z = kU$ for some unit normal vector U and some $k \in \mathbb{R}$, we have for the scalar 2nd fundamental form (with respect to U): $A(X,Y) = k\langle X, Y\rangle$. From the Codazzi equation (since $\bar{M}$ has constant curvature):

$$0 = X(A(Y,Z)) - Y(A(X,Z)) = X(k)\langle Y, Z\rangle - Y(k)\langle X, Z\rangle.$$

Thus $X(k)Y - Y(k)X = 0$ for all X, Y. In particular if $\langle X, Y\rangle = 0, Y \neq 0$, taking inner products with Y (non-null) yields $X(k) = 0$ for all X, so k is constant.

By the Gauss equation, for any non-degenerate 2-plane in TM with orthonormal basis $\{X, Y\}$, we have:

$$\sigma_{XY} = \bar{\sigma}_{XY} + \epsilon_N k^2 = \bar{C} + \epsilon_N k^2.$$

So M has constant curvature.

The converse also holds, if M has dimension at least three.

Proposition 2. If $M, \bar{M}$ have constant curvatures $C \neq \bar{C}$ and $dim(M) \geq 3$, then M is totally umbilic in $\bar{M}$.

Proof. Let $p \in M$. From the Gauss equation, we have for any $p \in M$, vectors $x, y, z \in T_pM$:

$$\langle S_p x, z\rangle S_p y - \langle S_p y, z\rangle S_p x = \epsilon_N (C - \bar{C})[\langle x, z\rangle y - \langle y, z\rangle x].$$

Let (e_a) be an orthonormal basis of T_pM diagonalizing S, $Se_a = \lambda_a e_a$. Using $x = e_a, y = e_b$, we find for each z (with $\Delta = \epsilon_N(C - \bar{C}) \neq 0$):

$$(\lambda_a \lambda_b - \Delta)\langle e_a, z\rangle e_b = (\lambda_a \lambda_b - \Delta)\langle e_b, z\rangle e_a,$$

and we can certainly choose z so that $\langle e_a, z\rangle, \langle e_b, z\rangle$ are both nonzero; so all the products $\lambda_a \lambda_b$ equal the same nonzero number Δ. Since $\dim(M) \geq 3$, this is only possible if all λ_a are equal, so S_p is a multiple of the identity, and M it totally umbilic.

Proposition 3. Let $M \subset \mathbb{M}^{n+1}$ be a connected, totally umbilic (but not totally geodesic) hypersurface in Minkowski space. Then M is an open set of a hyeperquadric (so if M is complete, M is a connected hyperquadric.)

Proof. From proposition 1 we know the shape operator w.r.t a unit normal U is $S = k\mathbb{I}$, for some constant k defined up to sign, depending on U (but $\frac{U}{k} = \frac{-U}{-k}$ is well-defined.) Define a map:

$$F : M \to \mathbb{M}^{n+1}, \quad F(p) = p + \frac{1}{k}U(p).$$

The differential is given by:

$$dF(p)[v] = v + \frac{1}{k}\bar{\nabla}_v U = v - \frac{1}{k}S_p v = 0, \quad v \in T_pM;$$

so $F \equiv const. := p_0 \in \mathbb{M}^{n+1}$, or $p - p_0 = -\frac{1}{k}U(p)$ for all $p \in M$. This implies:

$$\langle p - p_0, p - p_0 \rangle = -\frac{1}{k^2}\langle U, U \rangle = \pm\frac{1}{k^2},$$

the equation of a hyperquadric.

Corollary. If $M \subset \mathbb{M}^{n+1}$ is a connected hypersurface in Minkowski space with constant sectional curvature (and $dim(M) \geq 3$), and not totally geodesic, then M is isometric to an open subset of a hyperquadric.

Proof. Follows from the preceding propositions.

Remark. Note that the same proof shows that open sets of spheres are the (non-planar) hypersurfaces of constant curvature in $\mathbb{R}^{n+1}$, *provided* $n \geq 3$. The two-dimensional "pseudospheres" in $\mathbb{R}^3$ (with constant curvature -1) show the dimensional restriction is needed.

Exercise 3. **Anti-deSitter (AdS) spacetime.**

Denote by $\mathbb{M}_2^{n+2} = (\mathbb{R}^{n+2}, q)$ the vector space $\mathbb{R}^{n+2}$, with coordinates $\bar{x} = (x_1, \dots, x_n, u, v) = (x, u, v), x \in \mathbb{R}^n$, endowed with the quadratic form (non-degenerate, with index 2):

$$q(\bar{x}, \bar{x}) = \sum_{i=1}^{n} x_i^2 - u^2 - v^2 = |x|^2 - u^2 - v^2.$$

Consider the hyperquadric in $\mathbb{M}_2^{n+2}$:

$$M^{n+1} = \{\bar{x} | q(\bar{x}, \bar{x}) = -1\}.$$

(i) Find a diffeomorphism

$$F : \mathbb{R}^n \times S^1 \to M^{n+1}, \quad F(y, \theta) = (x(y, \theta), u(y, \theta), v(y, \theta)).$$

(ii) Compute the pullback metric F^*q of the metric induced by q on M (in coordinates (y, θ)), and show it is Lorentzian.

(iii) Show M^{n+1} is totally umbilic in $\mathbb{M}_2^{n+2}$, and describe its geodesics. Show M has closed timelike geodesics.

(iv) Use the Gauss formula to show M^{n+1} has constant *negative* sectional curvatures.

Remark. Anti deSitter (AdS) spacetime is the universal cover $\tilde{M}^{n+1}$, diffeomorphic to $\mathbb{R}^n \times \mathbb{R}$ and with no closed timelike geodesics.