

Math 567, fall 2008- Problems for lecture 1. (due 9/4)

1.,2. Recall the examples discussed in class of smooth injective immersions into \mathbb{R}^2 whose images are not 1-dimensional surfaces (say C^1) in \mathbb{R}^2 :

1. An injective immersion $f : \mathbb{R} \rightarrow \mathbb{R}^2$ with image a curve Γ_1 without self-intersections, so that for a point $p \in \Gamma_1$ we have $f(t) \rightarrow p$ as $t \rightarrow \pm\infty$.

2. An injective immersion $f : \mathbb{R} \sqcup S \rightarrow \mathbb{R}^2$ (disjoint union); S is the unit circle, f is the identity on S and the image Γ_2 of \mathbb{R} under f spirals towards D as $t \rightarrow \infty$.

Write down proofs that Γ_1 (in (1)) and $\Gamma_2 \sqcup S$ (in (2)) are not 1-dimensional surfaces in \mathbb{R}^2 .

3.[Berger-Gostiaux p. 98] Let H be the *hyperquadric* in \mathbb{R}^n with equation:

$$\sum_{1 \leq i \leq j \leq n} a_{ij} x_i x_j = 1,$$

where the symmetric matrix (a_{ij}) is invertible. Show that H :

(i) is a smooth $(n-1)$ -dimensional surface in \mathbb{R}^n ;

(ii) is diffeomorphic to $S^k \times \mathbb{R}^{n-1-k}$, where $(k, n-k)$ is the signature of the quadratic form defined by (a_{ij}) (k pluses, $n-k$ minuses).

4. Consider the group of $n \times n$ orthogonal real matrices:

$$O(n) = \{A \in M_n(\mathbb{R}); A^T A = I_n\}.$$

(i) Show that $O(n)$ is a smooth surface in some euclidean space, and compute its dimension;

(ii) Show that $O(n)$ is *compact*.

(iii) Show that $O(n)$ has two connected components. (*Hint*: First prove that any two *positive* orthonormal bases of \mathbb{R}^n can be connected by a path of positive orthonormal bases).