The local Gauss-Bonnet theorem.

Let Γ be a simple closed curve bounding the domain D in $(\mathbb{R}^2, g)$ (g is a Riemannian metric). Let $\{T, N\}$ be a positive g-orthonormal frame along Γ, with T tangent and N normal, pointing into D. The geodesic curvature of Γ is defined as:

$$k_g = \langle \frac{DT}{ds}, N \rangle, \text{ or } \frac{DT}{ds} = k_g N,$$

where $s \in [0, L]$ is the arc length parameter along Γ and D is covariant derivative in the L-C connection of g. Our goal is to compute:

$$\int_\Gamma k_g ds.$$

Let $\{e_1, e_2\}$ be a g-orthonormal frame in $\mathbb{R}^2$. Consider the 1-form:

$$\omega = \langle \nabla e_1, e_2 \rangle.$$

Then:

$$\int_0^L \langle \nabla_T e_1, e_2 \rangle ds = \int_0^L \omega(T) ds = \int_\Gamma \omega = \int_D d\omega,$$

by Stokes' theorem for line integrals.

A simple calculation shows:

$$k_g = \varphi'(s) + \langle \nabla_T e_1, e_2 \rangle,$$

where $\varphi(s)$ is defined by $T = (\cos\varphi) e_1 + (\sin\varphi) e_2$. Thus:

$$\int_\Gamma k_g ds = 2\pi + \int_\Gamma \omega,$$

so we need to compute $d\omega$.

Lemma. $d\omega = \langle R(e_1, e_2) e_1, e_2 \rangle \theta_1 \wedge \theta_2 = -K dA_g$.

Here K is the Gauss curvature of g and $\{\theta_1, \theta_2\}$ is the dual co-frame (so $\theta_1 \wedge \theta_2 = dA_g$ is the g-area form.)

Proof. We have:

$$\omega = f\theta_1 + g\theta_2, \quad f = \langle \nabla_{e_1} e_1, e_2 \rangle, \quad g = \langle \nabla_{e_2} e_1, e_2 \rangle.$$

Hence:

$$d\omega = [e_1(g) - e_2(f)]\theta_1 \wedge \theta_2 + f d\theta_1 + g d\theta_2,$$

where

$$d\theta_1(e_1, e_2) = e_1(\theta_1 e_2) - e_2(\theta_1 e_1) - \theta_1[e_1, e_2] = -\langle [e_1, e_2], e_1\rangle$$

and, likewise:

$$d\theta_2(e_1, e_2) = -\langle [e_1, e_2], e_2\rangle.$$

Thus:

$$fd\theta_1 = \langle \nabla_{e_1} e_2, [e_1, e_2]\rangle \theta_1 \wedge \theta_2, \quad gd\theta_2 = -\langle \nabla_{e_2} e_1, [e_1, e_2]\rangle \theta_1 \wedge \theta_2.$$

On the other hand:

$$e_1(g) = \langle \nabla_{e_1}(\nabla_{e_2} e_1), e_2\rangle,$$

since $\langle \nabla_{e_2} e_1, \nabla_{e_1} e_2\rangle$ vanishes ($\nabla_{e_2} e_1$ has no e_1 component and $\nabla_{e_1} e_2$ has no e_2 component.)

$$e_2(f) = \langle \nabla_{e_2}(\nabla_{e_1} e_1), e_2\rangle,$$

for a similar reason. We conclude:

$$d\omega = (\langle \nabla_{e_1}(\nabla_{e_2} e_1) - \nabla_{e_2}(\nabla_{e_1} e_1), e_2\rangle + \langle \nabla_{e_1} e_2 - \nabla_{e_2} e_1, [e_1, e_2]\rangle)\theta_1 \wedge \theta_2.$$

Now observe that:

$$\langle \nabla_{e_1} e_2 - \nabla_{e_2} e_1, [e_1, e_2]\rangle = \langle \nabla_{e_1} e_2 - \nabla_{e_2} e_1, \nabla_{e_1} e_2 - \nabla_{e_2} e_1\rangle$$
$$= \langle \nabla_{e_1} e_2, \nabla_{e_1} e_2\rangle + \langle \nabla_{e_2} e_1, \nabla_{e_2} e_1\rangle$$

(since the other pairings vanish, as seen above)

$$= -\langle \nabla_{e_1}(\nabla_{e_1} e_2), e_2\rangle - \langle \nabla_{e_2}(\nabla_{e_2} e_1), e_1\rangle := -T.$$

We *claim* that:

$$T = \langle \nabla_{[e_1,e_2]} e_1, e_2\rangle.$$

Since:

$$\langle \nabla_{[e_1,e_2]} e_1, e_2\rangle = \langle \nabla_{\nabla_{e_1} e_2} e_1, e_2\rangle + \langle e_1, \nabla_{\nabla_{e_2} e_1} e_2\rangle$$
$$= \langle \nabla_{e_1}(\nabla_{e_1} e_2), e_2\rangle - \langle [e_1, \nabla_{e_1} e_2], e_2\rangle + \langle \nabla_{e_2}(\nabla_{e_2} e_1), e_1\rangle + \langle [\nabla_{e_2} e_1, e_2], e_1\rangle,$$

the claim will follow from:

$$\langle [e_1, \nabla_{e_1} e_2], e_2\rangle = 0 = \langle [\nabla_{e_2} e_1, e_2], e_1\rangle.$$

To see this, write (for some function h):

$$\nabla_{e_1} e_2 = he_1.$$

Then:

$$[e_1, \nabla_{e_1} e_2] = [e_1, he_1] = e_1(h)e_1,$$

which has no e_2 component: $\langle [e_1, \nabla_{e_1} e_2], e_2 \rangle = 0$. The other equality is seen in the same way. Combining these remarks, we conclude:

$$\begin{aligned} d\omega &= (\langle \nabla_{e_1}(\nabla_{e_2} e_1) - \nabla_{e_2}(\nabla_{e_1} e_1), e_2 \rangle - \langle \nabla_{[e_1,e_2]} e_1, e_2 \rangle)\theta_1 \wedge \theta_2 \\ &= \langle R(e_1, e_2)e_1, e_2 \rangle \theta_1 \wedge \theta_2, \end{aligned}$$

as we wished to show.

This proves the *local Gauss-Bonnet formula:*

$$\int_\Gamma k_g ds + \int_D K dA_g = 2\pi.$$

Exercise. The set of equations:

$$d\theta_1 = \omega \wedge \theta_2, \quad d\theta_2 = -\omega \wedge \theta_1, \quad d\omega = -K\theta_1 \wedge \theta_2$$

is known as *Cartan's equations* for the frame $\{e_1, e_2\}$. Verify the other two equations.