SPACES OF CONTINUOUS FUNCTIONS

If X is a set and F a normed vector space, the set $B_F(X)$ of bounded functions from X to F, with the supremum norm, is also a normed vector space. (And a Banach space, if$F$ is a Banach space.)

For X a metric space, denote by $C^b_F(X) \subset B_F(X)$ the set of continuous bounded functions from X to F. With the sup norm that's also a normed vector space. (And a Banach space, if F is a Banach space.)

Proposition 1. $C^b_F(X)$ is a closed subspace of $B_F(X)$; that is, the uniform limit of continuous bounded functions is continuous.

Proof. Suppose $f_n \to f$ uniformly in X. Let $x_0 \in X$. Given $\epsilon > 0$, choose N large enough so that $\sup_X |f_N - f| \leq \epsilon$, then a neighborhood V of x_0 so that $|f_N(x) - f_N(x_0)| \leq \epsilon$ for all $x \in V$. Now note that, for each $x \in V$:

$$|f(x) - f(x_0)| \leq |f_N(x) - f(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \leq 3\epsilon.$$

It is easy to give examples of sequences of continuous functions converging *non-uniformly* to a continuous function. However, this can't happen for monotone sequences on compact spaces:

Dini's theorem. Let X be a compact metric space. If an increasing (or decreasing) sequence (f_n) of continuous real-valued functions on X converges to a continuous function f, then the convergence is uniform.

Proof. Given $\epsilon > 0$ and $x \in X$ we may find an integer $n(x)$ so that $f - f_{n(x)} \leq \epsilon$. By continuity, we may find a neighborhood $V(x)$ of x in X so that $|f(x) - f(y)| \leq \epsilon$ and $|f)n(x)(x) - f_{n(x)}(y)| \leq \epsilon$, for all $y \in V(x)$. Then for each $y \in V(x)$ we have $f(y) - f_{n(x)}(y) \leq 3\epsilon$. Take a finite sub cover of $\{V(x)\}_{x \in X}$ and the maximum N of the $n(x_i)$. Then for each $n \geq N$ we have $f(y) - f_N(y) \leq f(y) - f_{n(x_i)}(y) \leq 3\epsilon$, if $y \in V(x_i)$. Since the $V(x_i)$ cover X, this ends the proof.

Question. What goes wrong in this proof if the sequence is not monotone?

Stone-Weierstrass approximation theorem.

Let A be a vector space over R. A is an *algebra* (or: commutative algebra with unit) if there exists a 'multiplication operation' $A \times A \to A$, $(f, g) \mapsto f * g$ which is bilinear (linear in f and g), commutative ($f * g = g * f$) and there is an element $1 \in A$ (the 'unit') satisfying $f * 1 = 1 * f = f$ for all $f \in A$. A *subalgebra* of A is a vector subspace of A which is closed under the multiplication operation and contains the unit.

For any metric space X, the space $C^b_R(X)$ of continuous, bounded real-valued functions on X is an *algebra* over the field of real numbers (with the operation of pointwise multiplication), satisfying, for the uniform norm:

$$||fg|| \leq ||f||||g||.$$

The unit is the constant function 1.

Exercise 1. Use this to show that the closure $\bar{A}$ of any subalgebra of $C^b_R(X)$ is also a subalgebra. *Hint:* recall $\bar{A}$ is the set of functions in $C^b_R(X)$ which are uniform limits of functions in A. Given $f = \lim f_n, g = \lim g_n$, the main point is checking that $fg = \lim(f_n g_n)$. Estimate $|fg - f_n g_n|(x)$ in the natural way.

Definition. We say a sub algebra of $C^b_R(X)$ *separates points* if $\forall x \neq y$ in X we may find $f \in A$ with $f(x) \neq f(y)$.

Example. The polynomial functions in one variable (with real coefficients) form a subalgebra of $C^b_R([0,1])$. The polynomial functions in n real variables form a subalgebra of $C^b_R([0,1]^n)$. Both of these separate points.

The polynomials in one variable made up of even-degree monomials also form a subalgebra of $C^b_R([-1,1])$, which doesn't separate points (any such polynomial takes the same value at 1 and -1).

Interpolation property. Assume $A \subset C^b_R(X)$ is a subalgebra that separates points. For all $x \neq y$ in X and all real numbers a, b, there exists $f \in A$ so that $f(x) = a, f(y) = b$.

By assumption we know there exists $g \in A$ so that $g(x) \neq g(y)$. Set:

$$f = a + (b-a)\frac{g - g(x)}{g(y) - g(x)}.$$

(Note that adding a constant to an element of A yields another element of A, since the unit (the constant function 1) is in A.)

Stone-Weierstrass theorem. Let X be a compact metric space, $A \subset C_R(X)$ a subalgebra containing the constants and separating points. Then A is dense in the Banach space $C_R(X)$.

Main Lemma. The pointwise max and the pointwise min of finitely many functions in $\bar{A}$ is still in $\bar{A}$.

We first give the proof of the theorem assuming the main lemma, then prove the lemma. There are two steps:

Step 1. Given $f \in C_R(X), x \in X$ and $\epsilon > 0$, we find $g_x \in \bar{A}$ so that $g(x) = f(x)$ and $g_x(y) \leq f(y) + \epsilon$, for all $y \in X$.

Step 2. Using compactness, argue there are finitely many points $x_1, \ldots, x_N \in X$ so that $\varphi(x) = \max\{g_{x_1}(x), \ldots, g_{x_N}(x)\}$ (which is in $\bar{A}$, by the main lemma) satisfies:

$$f(y) - \epsilon \leq \varphi(y) \leq f(y) + \epsilon,$$

for all $y \in X$. Thus for any $\epsilon > 0$ we may find $g\varphi \in \bar{A}$ so that $||f - \varphi|| \leq \epsilon$ (uniform norm). So $f \in \bar{A}$.

Step 1. For each $f \in C_R(X)$, each $x \in X$ and any $\epsilon > 0$, there exists a function $g \in \bar{A}$ so that $g(x) = f(x)$ and $g(y) \leq f(y) + \epsilon \forall y \in X$.

Proof. Given $z \in X$ with $z \neq x$, let $h_z \in \bar{A}$ satisfy $h_z(x) = f(x)$ and $h_z(z) = f(z) + \epsilon/2$ (from the interpolation property.) By continuity, there is an open neighborhood V_z of z in X so that, for each $y \in V_z$ we have $h_z(y) \leq f(y) + \epsilon$. These define an open cover $\{V_z\}_{z \in X}$ of X. Taking a finite subcover $\{V_{z_i}\}_{i=1}^N$ of X, we find (from the Main Lemma) the function $g = \min\{h_{z_i} | i = 1, \ldots, N\}$ is in $\bar{A}$ and satisfies the conditions required.

Proof of Step 2.

Let $f \in C_R(X)$ be arbitrary Given $\epsilon > 0$ and $x \in X$, let $g_x \in \bar{A}$ be the function from Step 1. By continuity there is a neighborhood $U(x)$ of x in X so that $g_x(y) \geq f(y) - \epsilon$ for $y \in U(x)$. Cover X by a finite number of neighborhoods $U(x_i)$, $i = 1, \ldots, N$. Then (from the Main Lemma) the function $\varphi = \max(g_{x_i})$ is in $\bar{A}$ and satisfies $f(y) - \epsilon \leq \varphi(y) \leq f(y) + \epsilon$.

Proof of the Main Lemma.

Step 1. There exists a sequence (u_n) of real polynomials approximating $\sqrt{t}$ uniformly in $[0, 1]$.

Define u_n by recurrence, letting $u_1 = 0$ and setting:

$$u_{n+1}(t) = u_n(t) + \frac{1}{2}(1 - u_n(t)^2).$$

We show by induction that $u_{n+1} \geq u_n$ and $u_n(t) \leq \sqrt{t}$ in $[0, 1]$. It follows from the recursion relation that the first fact follows from the second. On the other hand,

$$\sqrt{t} - u_{n+1}(t) = \sqrt{t} - u_n(t) - \frac{1}{2}(t - u_n^2(t)) = (\sqrt{t} - u_n(t))(1 - \frac{1}{2}(\sqrt{t} + u_n(t))$$

and from $u_n(t) \leq \sqrt{t}$ it follows that the second factor is positive. Thus we have pointwise convergence of u_n to $\sqrt{t}$ (from the recurrence relation), and then uniform convergence follows from Dini's theorem.

Step 2. If $f \in \bar{A}$, then $|f| \in \bar{A}$, the closure of A in $C_R(X)$.

Let $a = ||f||$ (sup norm). The function f^2/a^2 is in $\bar{A}$ (since $\bar{A}$ is an algebra) and takes values in $[0,1]$. If $u_n(t)$ are the functions from Step 1, the compositions $u_n \circ (f^2/a^2)$ are in $\bar{A}$ (again since A is an algebra), and converge uniformly to $|f|/a$.

Step 3. If $f, g \in A$, then $\max\{f,g\}, \min\{f,g\}$ are also in $\bar{A}$.

$$\max\{f,g\} = \frac{1}{2}(f+g+|f-g|), \quad \min\{f,g\} = \frac{1}{2}(f+g-|f-g|).$$

Step 4 (final). The pointwise max and the pointwise min of finitely many functions in $\bar{A}$ is still in $\bar{A}$: follows from Step 3, since minimizing over a finite set amounts to a finite number of pair comparisons.

Remark. The theorem is false for subalgebras of $C_{\mathbb{C}}(X)$ (complex-valued functions). This follows from the classical result in Complex Analysis:

Let $f_n : D \to \mathbb{C}$ be a sequence of complex analytic functions in a domain $D \subset \mathbb{C}$. Suppose $f_n \to f$ uniformly on compact subsets of D. Then $f : D \to \mathbb{C}$ is analytic in D.

Corollary 1. (Polynomials in R^n.) Any real-valued continuous function on a compact subset of R^n is the uniform limit of a sequence of polynomials.

Corollary 2. (Separability.) If X is a compact metric space, the space $C_R(X)$ is separable.

Let (U_n) be a countable basis for the topology of X, and let $g_n(x) = d(x; X \setminus U_n)$. The monomials $g_1^{m_1} \dots g_r^{m_r}$ (with the m_j integers) form a countable set (h_n) of continuous functions on X, and the vector space they span is the algebra A generated by the g_n. So it suffices to use the Stone-Weierstrass theorem to conclude A is dense in $C_R(X)$.

The family $\{g_n\}$ separates points: if $x \neq y$, we may find an U_n so that $x \in U_n, y \notin U_n$, and thus $g_n(x) \neq 0, g_n(y) = 0$.

Alternative approach: approximation by Bernstein polynomials.

Cultural Remark 2: Neural Networks.

Remark. A metric space X is *precompact* if for all $\epsilon > 0$ a finite number of balls with diameter ϵ suffices to cover X. The following three conditions are equivalent for a metric space: (i) X is compact; (ii) X is sequentially compact (any infinite sequence has an accumulation point.) (iii) X is precompact and complete.

A precompact metric space X is separable. Indeed for each n there is a finite set A_n so that, for each $x \in X$, $d(x, A_n) \leq 1/n$. Then $A = \cup_n A_n$ is countable, and it is easy to see that $\bar{A} = X$.

Equicontinuous sets. Let X be a metric space, E a normed vector space. A family (subset) $H \subset B_E(X)$ of the set of bounded functions $f : X \to E$ is *equicontinuous* at $x_0 \in X$ if $(\forall \epsilon > 0)(\exists \delta > 0)(\forall f \in H)(d(x, x_0) \leq \delta \Rightarrow |f(x) - f(x_0)| \leq \epsilon)$.

H is *equicontinuous* if it is equicontinuous at every point of X. For example, families of functions satisfying a uniform Lipschitz (or Hölder) condition on X are equicontinuous.

1. If $(f_n) \subset B_E(X)$ converges pointwise in X to a function $g \in B_E(X)$, and is equicontinuous at $x_0 \in X$, then g is continuous at x_0.

2. If E is a Banach space, $(f_n) \subset C^b_E(X)$ an equicontinuous sequence, and $f_n(x) \to g(x)$ pointwise for x in a dense subset $D \subset X$, then $f_n \to g$ pointwise on X. (Since E is complete, it suffices to show (f_n) is Cauchy on X.)

3. If X is compact metric, $(f_n) \subset C_E(X)$ is equicontinuous and $f_n \to g$ pointwise in X, then $f_n \to g$ uniformly in X.

Ascoli's theorem. Let X be a compact metric space, E a Banach space. For a family $H \subset C_E(X)$ to be relatively compact, the following conditions are necessary and sufficient: (i) H is equicontinuous; (ii) for each $x \in X$, the set $H(x) = \{f(x); f \in H\}$ is relatively compact in E. (That is, its closure is compact; if E is finite-dimensional, this just means $H(x)$ is bounded, for each $x \in X$.)

Remark. A relatively compact set is precompact; the converse is true in a complete metric space.

Proof of Ascoli's theorem.

(i) Necessity. If H is relatively compact, for each $\epsilon > 0$ there exist a finite number of functions $f_i \in H$ so that for each $f \in H$ we have $||f - f_i|| \leq \epsilon||$ (sup norm), for some i. Thus for each $x \in X$ we have $|f(x) - f_i(x)| \leq \epsilon$, so the set $H(x) \subset E$ is precompact, and (since E is complete) also relatively compact.

Let $x \in X$ and $V(x)$ be a neighborhood of x so that $|f_i(x) - f_i(y)|$ for each $y \in V(x)$, and all i. If follows that if $y \in V(x)$ we have $|f(x) - f(y)| < 3\epsilon$ for each $f \in H$, so H is equicontinuous.

(ii) Sufficiency. Since $C_E(X)$ is complete, it suffices to show H is precompact.

Given $\epsilon > 0$, for each $x \in X$ let $V(x)$ be a neighborhood so that $|f(x) - f(y)| < \epsilon$, for each $f \in H$ and $y \in V(x)$. Take a finite sub covering $V(x_i), i = 1, \ldots, n$. The set $J = \cup_{i=1}^{n}\{f(x_i); f \in H\}$ is (by hypothesis) relatively compact, hence precompact; so there is a finite subset $K = \{c_1, \ldots, c_m\}$ of J so that each element of J is ϵ-close to some element of K.

We have to exhibit a covering of H by finitely many subsets of diameter at most ϵ. For each function $\varphi : \{1, \ldots, n\} \to \{1, \ldots, m\}$ let L_φ be the set of all $f \in H$ such that $|f(x_i) - c_{\varphi(i)}| < \epsilon$. Some sets L_φ may be empty, but they certainly cover H, and there are only finitely many of them. We have to check the L_φ have small diameter.

Let $f, g \in L_\varphi$, for some function φ as above. Then if $y \in X$, there is an $x_i, i = 1, \ldots n$ so that $y \in V(x_i)$, so $|f(y) - f(x_i)| \leq \epsilon$ and $|g(y) - g(x_i)| < \epsilon$. On the other hand, $f(x_i) - c_{\varphi(i)}| \leq \epsilon$ and $|g(x_i) - c_{\varphi(i)}| \leq \epsilon$; so in the end we have $|f(y) - g(y)| \leq 4\epsilon$ for each i, or $||f - g|| \leq 4\epsilon$ (sup norm.)