DEFINITIONS AND THEOREMS IN GENERAL TOPOLOGY

1. Basic definitions.

A *topology* on a set X is defined by a family $\mathcal{O}$ of subsets of X, the *open sets* of the topology, satisfying the axioms: (i) $\emptyset$ and X are in $\mathcal{O}$; (ii) the intersection of finitely many sets in $\mathcal{O}$ is in $\mathcal{O}$; (iii) arbitrary unions of sets in $\mathcal{O}$ are in $\mathcal{O}$.

Alternatively, a topology may be defined by the neighborhoods $U(p)$ of an arbitrary point $p \in X$, where $p \in U(p)$ and, in addition:

(i) If U_1, U_2 are neighborhoods of p, there exists U_3 neighborhood of p, such that $U_3 \subset U_1 \cap U_2$;

(ii) If U is a neighborhood of p and $q \in U$, there exists a neighborhood V of q so that $V \subset U$.

A topology is *Hausdorff* if any distinct points $p \neq q$ admit disjoint neighborhoods. This is almost always assumed.

A set $C \subset X$ is *closed* if its complement is open. The closure $\bar{A}$ of a set $A \subset X$ is the intersection of all closed sets containing X. A subset $A \subset X$ is *dense* in X if $\bar{A} = X$.

A point $x \in X$ is a *cluster point* of a subset $A \subset X$ if any neighborhood of x contains a point of A distinct from x. If A' denotes the set of cluster points, then $\bar{A} = A \cup A'$.

A map $f : X \to Y$ of topological spaces is *continuous at* $p \in X$ if for any open neighborhood $V \subset Y$ of $f(p)$, there exists an open neighborhood $U \subset X$ of p so that $f(U) \subset V$. f is *continuous* if it is continuous at every point; equivalently, if the *preimage* $f^{-1}(V)$ of each open set $V \subset Y$ is open in X.

A family $\mathcal{F}$ of open sets is a *basis* for a topology if any open set is a union of sets in $\mathcal{F}$; a *local basis* at a point $p \in X$ if any neighborhood of p is contained in a set in $\mathcal{F}$.

A topology is *second-countable* if it admits a countable basis; *first-countable* if it has a countable local basis at each point; *separable* if X has a countable dense set. Second-countable spaces are separable, but not conversely.

A topological vector space (TVS) is a vector space E (over $\mathbb{R}$, say) endowed with a topology so that (i) $\{0\}$ is a closed set (this guarantees Hausdorff); (ii) the vector space operations $(x, y) \mapsto x + y$ and $(\lambda, x) \mapsto \lambda x$ are continuous.

Hierarchy of separation properties. Not every Hausdorff space admits non-constant continuous functions to the unit interval. A Hausdorff space X is *completely regular* if given $p \in X, A \subset X$ closed with $p \notin A$, there exists a continuous function $f : X \to [0,1]$ so that $f(p) = 0$ and $f \equiv 1$ in A. A weaker condition is *regular*: p, A as before admit disjoint open neighborhoods.

X (Hausdorff) is *normal* if any disjoint closed sets admit disjoint open neighborhoods. Equivalently (Urysohn) given $A, B \subset X$ disjoint closed sets, there exists $f : X \to [0,1]$ continuous so that $f \equiv 0$ in A, $f \equiv 1$ in B. Thus normal spaces are completely regular.

To get $f^{-1}(0) = A$ we must require A to be a G_δ: a countable intersection of open sets. A normal space in which each closed set is a G_δ is called *perfectly normal.*

Another equivalent condition to normality (Tietze) is that given $A \subset X$ closed and $f : A \to \mathbb{R}$ continuous with $|f(a)| \leq M$ in A, there exists a continuous extension of f to X, satisfying $|f(x)| \leq M$ in X.

Covering properties. A Hausdorff space is *paracompact* if every open covering has an open locally finite finite *refinement*.

Thm. Every paracompact space is normal. [D, p. 163]

Thm. Let Y be paracompact. For each open covering of Y, there is a partition of unity subordinate to it.[D, p. 170]

A Hausdorff space is *Lindelöf* if every open covering contains a countable *subcovering.* (Note that a subcovering is a very special type of refinement.)

Lindelöf's theorem. Every second-countable space is Lindelöf. (The converse is false.)[D, p. 174]

Morita's theorem. Regular+Lindelöf $\Rightarrow$ paracompact. [D, p. 174]

Theorem. Separable + paracompact $\Rightarrow$ Lindelöf. [D, p. 177]

Definition. A subset $A \subset X$ of a Hausdorff space is *compact* if any open covering of A admits a finite subcovering. Clearly every compact space is paracompact (hence also normal). It is easy to see that if $f : X \to Y$ is continuous and $A \subset X$ is compact, then $f(A)$ is compact in Y.

A metric space (or a normed vector space) is *complete* if Cauchy sequences converge. A *Banach space* is a complete normed vector space. A *Fréchet space* is a complete, locally convex vector space with a translation-invariant metric.

2. Metric spaces. Every metric space is *perfectly normal*: normal, and each closed set is a G_δ (i.e. a countable intersection of open sets.) [D, p.

186] (Just take the distance function to a closed set).

Every metric space is first-countable. (The open balls at p with rational radius give a local basis at p.)

A.H.Stone's theorem. Every metric space is paracompact. [D, p. 186]

Theorem. A metric space is second countable *iff* it is separable *iff* it is Lindelöf. [D, p. 186]

Nagata-Smirnov therem. A topological space is metrizable if and only if it is regular and admits a basis which can be decomposed into a countable union of locally finite families of open sets. [D, p. 194]

A consequence is *Urysohn's theorem.* Any second-countable regular space is metrizable.[D, p. 195]

A topological vector space E is metrizable if (and, as is clear, only if) E is first-countable. In this case the metric can be taken invariant under addition ($d(x+z, y+z) = d(x, y)$) and open balls at the origin *balanced*: $x \in B, |\lambda| \leq 1 \Rightarrow \lambda x \in B$. If X is *locally convex* (admits a local basis at 0 consisting of convex sets), then d can be taken to satisfy *open balls are convex.* [R, p. 18]

Remark. In a TVS E with topology given by a (translation-invariant) metric d, there are two notions of "bounded" (for a subset $A \subset E$): the usual one defined by the metric, and the following: for each open neighborhood V of 0 in E , one may find $\lambda \in \mathbb{R}$ so that $A \subset \lambda V$. If $A \subset E$ is bounded in this sense, it is d-bounded (*exercise*) , but not conversely. (Indeed the metric $d_1 = d/(1+d)$ defines the same topology, and the whole space E is d_1-bounded). Usually for a general TVS one takes the more general (stronger) definition of 'bounded'.

A topological vector space is *normable* (topology may be given by a norm) iff it is locally convex and locally bounded (i.e. 0 has a bounded neighborhood U , meaning for every neighborhood V of 0 there exists a $t > 0$ so that $U \subset sV$ for all $s > t$.) [R, p. 28]. Note that 'locally bounded' implies 'first countable' (for topological vector spaces): if V is a bounded neighborhood of 0 and $r_n \to 0$, $\{r_n V\}$ gives a countable basis at 0. [R, p.15]

3. Convergence. A sequence $(x_n)_{n\geq 1}$ in a space Y *converges* to $y_0 \in Y$ if $\forall U(y_0)$ the (x_n) are eventually in $U(y_0)$; the sequence *accumulates* at y_0 if $\forall U(y_0)$ the x_n are in $U(y_0)$ for arbitrarily large n.

Adequacy of sequences. Let Y be (Hausdorff and) first countable. Then a sequence (x_n)in Y accumulates at y_0 iff some subsequence converges to y_0.

Let Y be first-countable, $A \subset Y$. Then $x \in \bar{A}$ iff there exists a sequence in A converging to x. (Recall $\bar{A} = \{x | \forall U(x), U(x) \cap A \neq \emptyset\}$.)

If X is first-countable, $f : X \to Y$ continuous (Y arbitrary), f is continuous at x_0 iff $f(x_n) \to f(x_0)$, for all sequences $x_n \to x_0$.

4. Compactness. *Def.* A Hausdorff space is *compact* if every open covering admits a finite subcovering. Clearly every compact space is paracompact (hence also normal.) It is easy to show that the image of a compact set under a continuous map is compact.

A compact subset A of a Hausdorff space X is always closed. Indeed if $x_0 \notin A$, for each $a \in A$ find disjoint neighborhoods $U(a), U_a(x_0)$. Taking a finite subcover $\{U(a_i)\}$, we have $\bigcup_i U(a_i)$ and $\bigcap_i U_{a_i}(x_0)$ are disjoint open neighborhoods of A and x_0.

Theorem. If Y is a compact space and $f : Y \to \mathbb{R}$ is continuous, then f attains its supremum and its infimum, and both are finite [D, p. 227].

A corollary of this is the fact that in a metric space (Y, d), any compact subset A is (closed and) bounded. Just apply the theorem to the function $d(a, a_0)$, where $a_0 \in A$.

In $\mathbb{R}^n$ we have the *Heine-Borel property*: A closed and bounded subset of $\mathbb{R}^n$ is compact.

If a *locally bounded* topological vector space E has the Heine-Borel property, then it is locally compact, hence finite-dimensional. (See below). Indeed, $0 \in E$ has a bounded neighborhood V, and its closure $\bar{V}$ is also bounded (true in any TVS), hence compact if we assume Heine-Borel holds.

However, not every TVS with the Heine-Borel property is finite-dimensional. An example [R, p.33] is the space $C^\infty(\Omega)$, ($\Omega \subset \mathbb{R}^n$ open): the topology of uniform convergence of finitely many derivatives on compact subsets $K \subset \Omega$ is metrizable, and the resulting Fréchet space structure does have the Heine-Borel property. (Hence it is not locally bounded.)

Lebesgue number. In a compact metric space (Y, d), if $\{U_\alpha\}$ is an open covering, there exists a number $\lambda > 0$ so that each ball $B(y, \lambda)$ is contained in some U_α [D, p. 234]. A consequence is the fact that continuous maps from compact metric spaces are uniformly continuous.

Def. A Hausdorff space is *sequentially compact* if every *countable* open covering admits a finite subcovering.

Clearly compact implies countably compact; but not conversely.

Theorem. The following are equivalent (for a Hausdorff space Y) [D, p.229]:

(1) Y is countably compact;

(2) Every countably infinite set subset $S \subset Y$ has at least one cluster point y_0 (i.e. $\forall U(y_0)$, $U(y_0) \cap S \neq \emptyset$.)

(3) Every sequence in Y has an accumulation point (Bolzano-Weierstrass property.)

Theorem. (paracompact or Lindelöf)+sequentially compact$\Rightarrow$ compact.[D, p.230]

Theorem. first-countable+ countably compact$\Rightarrow$ regular. [D, p.230]

Local compactness. A Hausdorff space is *locally compact* if each point has a relatively compact neighborhood (its closure is compact). In a TVS, it is enough to require that 0 has a relatively compact neighborhood.

Every locally compact space is completely regular [D, p. 239]

Any locally compact TVS is finite-dimensional [R, p.18].

Theorem. [D, p.241] The following conditions are equivalent:

(1) Y is Lindelöf and locally compact;

(2) Y is a countable union of compact spaces (i.e. σ-*compact*);

(3) Y admits an increasing exhaustion by countably many relatively compact open subsets: $Y = \bigcup_i U_i$ with $\bar{U}_i \subset U_{i+1}$.

Baire's theorem. In a locally compact space, the intersection of a countable family of open dense sets is dense. [D, p. 249].

5. Completeness and Category.

Def. A subset $A \subset X$ of a metric space is *precompact* (or *totally bounded*) if, for any $\epsilon > 0$, A can be covered by finitely many balls of radius at most ϵ.

Theorem. A metrizable space Y is compact iff it has a metric that is both complete and precompact. A subset $A \subset Y$ of a complete metric space has compact closure iff it is totally bounded. [D, p. 298].

Baire's Theorem. Any complete metric space has the Baire property: a countable intersection of open dense sets is dense.

Remark: Every locally compact metric space admits a (possibly different) complete metric. Furthermore, if Y is compact, every metric on Y is complete. [D. p. 294]

REMARK: For a summary, see the diagram in [D, p. 311].