LOCALLY COMPACT BANACH SPACES ARE FINITE DIMENSIONAL

1. Hyperplanes and bounded linear functionals.

Basic fact. A linear map $L : E \to F$ of normed vector spaces is continuous iff there exists an $M > 0$ so that $|Lv|_F \leq M|v|_E$, for all $v \in E$. (We refer to this by saying 'L is bounded'.)

Proof. If such an M exists, we have $|Lv_1 - Lv_2|_F \leq M|v_1 - v_2|_E$. Conversely, if L is continuous, there is an $r > 0$ so that $|v|_E \leq r \Rightarrow |Lv|_F \leq 1$ Then for any nonzero $v \in E$ we have $|L(\frac{rv}{|v|_E})|_F \leq 1$, or $|Lv|_F \leq \frac{1}{r}|v|_E$.

Remark: Any linear map $L : R^m \to R^n$ is automatically bounded: for $x = (x^1, \dots x^m) = \sum_{i=1}^m x^i e_i$, we have:

$$||L(x)|| \leq \sum_{i=1}^{m} |x_i| ||L(e_i)|| \leq M \sum_i |x_i| \leq M\sqrt{m}||x||, \text{ where } M = \max_{1 \leq i \leq m} |L(e_i)|,$$

and $||x||$ denotes the euclidean norm.

Hyperplanes. A it hyperplane is a subspace $H \subset E$ of a (real) vector space E satisfying, for each $a \notin H$:

$$E = H \oplus \mathbb{R}a$$

(algebraic direct sum).

A 'linear functional' on a vector space E is a linear map $f : E \to R$. There is a general correspondence between hyperplanes and linear functionals. First, if f is a linear functional, $H = ker(f)$ (the kernel, or nullspace of f) is a hyperplane.

Exercise 1. Prove this.

In the other direction: given a hyperplane H and a nonzero vector $a \notin H$, define a linear functional $f : E \to R$ by $x = h + f(x)a$, where $h \in H$. This is the unique linear functional with $ker(f) = H$ and $f(a) = 1$. Conversely, given a nonzero linear functional $f : E \to R$, the subspace $H = Ker(f)$ is a hyperplane: fix $a \in E$ so that $f(a) \neq 0$; then for each $x \in E$, $h(x) = x - \frac{f(x)}{f(a)}a$ is in H (and $H \cap \mathbb{R}a = \{0\}$), so $E = H \oplus \mathbb{R}a$.

Proposition 1. Let E be a normed vector space, $H \subset E$ a hyperplane, $f : E \to R$ any linear functional with $H = ker(f)$. H is closed in E iff f is a continuous linear functional.

Proof. That $H = \ker(f)$ is closed if f is continuous is clear, since $H = f^{-1}(\{0\}$.

Conversely, assume H is closed and let $a \in E$ be such that $f(a) = 1$.

Exercise 2. Show that if $f : E \to R$ is a linear functional and $f(a) = 1$, then $\{w \in E | f(w) = 1\} = a + H$, where $H = ker(f)$ and we define $a + H = \{v \in E | v = a + h \text{ for some } h \in H\}$.

Since $a + H$ is closed and does not contain $0 \in E$, there is a ball $B = \{v \in E, |v|_E \leq r\}$ disjoint from $a + H$. That is:

$$|v|_E \leq r \Rightarrow f(v) \neq 1.$$

We *claim* that, in fact:

$$|v|_E \leq r \Rightarrow |f(v)| \leq 1$$

.

If not, evidently (for such a $v \in B$) $f(v) \neq 0$, and letting $w = \frac{v}{f(v)}$ we see that $|f(v)| > 1$ implies $w \in B$, while $f(w) = 1$, so w is in $a + H$ and also in B, contradiction. This proves the claim, and (by considering $rv/|v|_E$, which is in B) the claim shows that, for all $v \in E$: $|f(v)| \leq (1/r)|v|_E$, so f is a bounded linear functional.

2. Finite-dimensional subspaces are closed.

Proposition 2. If V is a closed subspace, W a finite-dimensional subspace of a normed vector space E, then $V + W$ is closed in E. In particular, any finite-dimensional subspace is closed.

Proof. First, one-dimensional subspaces are closed: let $(v_n)_{n \geq 1}$ be a sequence of the form $v_n = \lambda_n a$, where $\lambda_n \in R$ and $a \neq 0$ is a fixed vector in E. Suppose $v_n \to b \in E$. Then (v_n) is a Cauchy sequence, and since $|\lambda_n - \lambda_n| = \frac{1}{|a|_E}|v_n - v_m|_E$, it follows that (λ_n) is a Cauchy sequence, so $\lambda_n \to \lambda_0$, and $v_n \to \lambda_0 a$. This shows $b = \lambda_0 a$.

In general, by induction on the dimension of W, it suffices to consider the case $W = \mathbb{R}a$, $a \notin V$. Then any $x \in V + W$ can be written uniquely in the form $x = f(x)a + y$ with $y \in V$, for a linear functional $f : V + W \to \mathbb{R}$, which is continuous since V is a closed hyperplane in $V + W$.

Exercise 3. Show that $f : V \oplus \mathbb{R}a \to \mathbb{R}$ defined by $x = y + f(x)a$ (where $y \in V$) is a linear functional on $V \oplus \mathbb{R}a$. *Hint:* uniqueness of this representation of x.

Let $b \in E$ be in the closure of $V + W$, $b = \lim x_n, x_n = f(x_n)a + y_n$. Since f is bounded, $f(x_n)$ is a Cauchy sequence in $\mathbb{R}$, converging to $\lambda \in \mathbb{R}$. Thus y_n has the limit $b - \lambda a$, which is in V (since V is closed). Thus $b \in V + W$.

Alternative proof.[1] Let E be a normed vector space, $V \subset E$ a finite-dimensional subspace. Let $(x_n)_{n\geq 1}$ be a sequence of vectors in V, converging to $x_0 \in E$ in the norm of E. In particular, (x_n) is a Cauchy sequence in E.

The norm of E induces a norm in V, which (since V is finite-dimensional) is equivalent to the Euclidean norm, and therefore complete. Thus (x_n) (being a Cauchy sequence in V) converges to a vector $v_0 \in V$. By uniqueness of limits (or directly via the triangle inequality) we see that $v_0 = x_0$, and therefore $x_0 \in V$. This shows V is closed in E.

3. Distance to a closed subspace.

Let $V \subset E$ be a subspace of a normed vector space E. Given $x \in E$, we define the distance to V by:

$$d(x, V) = \inf\{|x - y|; y \in V\}.$$

If E is a Banach space, V is a proper closed subspace and $x \in E \setminus V$, then $d(x, V) > 0$. For if $d(x, V) = 0$ we may take a sequence $y_n \in V$ with $|x - y_n| \to 0$. The triangle inequality implies (y_n) is a Cauchy sequence, converging (in view of the hypotheses) to $y \in V$ with $|x - y| = 0$, contradiction.

Exercise 4. Show that if V is locally compact (in particular, if V is finite dimensional) this infimum is attained by some vector in V (that is, $(\exists v \in V) d(x, V) = |x - v|$).

Hint. Let (y_n) be a minimizing sequence in V. Use the triangle inequality to show (y_n) is bounded.

Lemma 1. Let E be a Banach space, $V \subset E$ a proper closed subspace. Then for every $\epsilon > 0$ there exists $x_0 \in E$ with $|x_0| = 1$ and $|x_0 - x| \geq 1 - \epsilon$, $\forall x \in V$.

Proof. Let $x' \in E \setminus V$, $d = dist(x', V) > 0$ (since V is closed), $\eta > 0$ arbitrary. Then there is $y' \in V$ so that $d \leq |x' - y'| \leq d + \eta$. Let $x_0 = \frac{x'-y'}{|x'-y'|}$. For any $x \in V$ we have:

$$|x_0 - x| = \frac{1}{|x' - y'|}|x' - y' - |x' - y'|x| \geq \frac{d}{|x' - y'|} \geq \frac{d}{d + \eta} = 1 - \epsilon \text{ if } \eta = \frac{\epsilon d}{1 - \epsilon},$$

since $y' + |x' - y'|x \in V$.

4. Locally compact Banach spaces are finite-dimensional.

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[1] This proof was shown to me by Ryan Unger, a student in Math 447-fall 2016.

Definition. A normed vector space space E is *locally compact* if any bounded subset of E has compact closure in E. (In particular, in this case the unit sphere is compact.) Equivalently, any bounded sequence in E has a convergent subsequence.

By the Heine-Borel theorem, finite-dimensional Banach spaces are locally compact. The converse is true.

Theorem. (S. Banach) Any locally compact Banach space E is finite dimensional.

Proof. Let $x_1 \in E$ be arbitrary, with unit norm. Given $x_1, \ldots, x_r$ in E of unit norm, let $G_r \subset E$ be the r-dimensional subspace of E spanned by these vectors. Being finite-dimensional, G_r is a closed subspace of E. If it is a proper subspace, by the lemma we may find a unit vector $x_{r+1} \in E$ with $|x_{r+1} - x_i| \geq 1/2, i = 1, \ldots, r$.
If we may do this for each r, we obtain an infinite sequence $(x_r)_{r \geq 1}$ of unit vectors satisfying $|x_p - x_q| \geq 1/2$ for each $p \neq q$, in particular admitting no convergent subsequence. This contradicts the assumption that E is locally compact.

Separable normed spaces. A metric space is *separable* if it has a countable dense set (equivalently, a countable basis of open sets.)

A sequence $(a_n)_{n \geq 1}$ of vectors in a normed vector space E is a *total sequence* if the set of finite linear combinations of vectors in the sequence is dense in E.

Proposition. A normed vector space is separable iff it has a total sequence consisting of linearly independent vectors.

Proof. (Outline.) If E has a total sequence (a_n), it is easy to see that the set D of all finite linear combinations of the a_n with rational coefficients is countable, and also dense, since:

$$|(\lambda_1 a_1 + \ldots \lambda_N a_N) - (r_1 a_1 + \ldots + r_N a_N)| \leq \sum_{j=1}^{N} |\lambda_j - r_j||a_j|.$$

Conversely, if E is separable (and infinite-dimensional), let $(a_n)_{n \geq 1}$ be a countable dense set, where each $a_n \neq 0$. Given $a_{n_1} = a_1, \ldots a_{n_k}$, let $a_{n_{k+1}}$ be the first vector on the list which is not in the linear span of $a_{n_1}, \ldots a_{n_k}$. Such a vector must exist, since (as seen in Lemma 2) such a span is closed, and is not E. It is easy to see that the sequence $(a_{n_k})_{k \geq 1}$ is a total sequence of l.i. vectors.

Exercise. The spaces c_0 of sequences of real numbers with limit zero (with the *sup* norm) and l^1 of summable sequences (with the norm $\sum_j |x_j|$) are separable Banach spaces, while the space l^∞ of bounded sequences (with the sup norm) is complete, but not separable.