INVARIANCE OF THE LAPLACE OPERATOR.

The goal of this handout is to give a coordinate-free proof of the invariance of the Laplace operator under orthogonal transformations of $\mathbb{R}^n$ (and to explain what this means).

Theorem. Let $D \subset \mathbb{R}^n$ be an open set, $f \in C^2(D)$, and let $U \in O(n)$ be an orthogonal linear transformation leaving D invariant ($U(D) = D$). Then:

$$\Delta(f \circ U) = (\Delta f) \circ U.$$

Before giving the proof we recall some basic mathematical facts.

Orthogonal linear transformations. An invertible linear transformation $U \in L(\mathbb{R}^n)$ is *orthogonal* if $U^*U = UU^* = \mathbb{I}_n$ (equivalently, $U^{-1} = U^*$.) Recall that, given $A \in L(\mathbb{R}^n)$, $A^* \in L(\mathbb{R}^n)$ is defined by:

$$A^*v \cdot w = v \cdot Aw, \quad \forall v, w \in \mathbb{R}^n.$$

Orthogonal linear transformations are *isometries* of $\mathbb{R}^n$; they preserve the inner product:

$$Uv \cdot Uw = v \cdot w, \quad \forall v, w \in \mathbb{R}^n, \text{ if } U \in O(n),$$

and therefore preserve the lengths of vectors and the angle between two vectors. Orthogonal linear transformations form a *group* (inverse of orthogonal is orthogonal, composition of orthogonal is orthogonal), denoted by $O(n)$. The matrix of an orthogonal transformation with respect to an orthonormal basis of $\mathbb{R}^n$ is an orthogonal matrix: $U^tU = \mathbb{I}_n$, so that the columns of U (or the rows of U) give an orthonormal basis of $\mathbb{R}^n$.

$\det(U) = \pm 1$ if $U \in O(n)$. The orthogonal transformations with determinant one can be thought of as rotations of $\mathbb{R}^n$. If $n = 2$, they are exactly the rotation matrices R_θ:

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

If $n = 3$, one is always an eigenvalue, with a one-dimensional eigenspace (the *axis* of rotation); in the (two-dimensional) orthogonal complement of this eigenspace, U acts as rotation by θ.

Change of variables in multiple integrals. Let $A \in L(\mathbb{R}^n)$ be an invertible linear transformation. Then if $D \subset \mathbb{R}^n$ and $f : A(D) \to \mathbb{R}$ is integrable, then so is $f \circ A$ and:

$$\int_{A(D)} f dV = \int_D (f \circ A)|det A| dV.$$

(This is a special case of the change of variables theorem.)

Multivariable integration by parts. Let $D \subset \mathbb{R}^n$ be a bounded open set with C^1 boundary, $f \in C^2(D)$, $g \in C_0^1(D)$. Then:

$$\int_D (\Delta f) g dV = -\int_D \nabla f \cdot \nabla g dV.$$

Here $C_0^1(D)$ denotes the set of C^1 functions in D which are zero near the boundary of D. This follows directly from *Green's first identity*, which in turn is a direct consequence of the divergence theorem.

Note that given $D \subset \mathbb{R}^n$ open and $P \in D$, it is easy to find a function $\varphi \in C_0^1(D)$ so that $\varphi \geq 0$ everywhere and $\varphi(P) = 1$. For example, one could let:

$$\varphi_P(x) = \max\{0, 1 - ||P - x||^2/\epsilon^2\}.$$

This works for $\epsilon > 0$ small enough, since φ_P is positive only in a ball of radius ϵ centered at P (which does not touch the boundary of D if ϵ is small.)

This can be used to observe that if h is continuous in D, then $h \equiv 0$ in D if, and only if, for any $\varphi \in C_0^1(D)$ we have:

$$\int_D h\varphi dV = 0.$$

Indeed if $h(P) \neq 0$ for some $P \in D$ we have (say) $h(P) > 0$, hence $h > 0$ in a sufficiently small ball B_ϵ centered at P (by continuity of h). But then $h\varphi_P \geq 0$ everywhere in D and yet (since $\varphi_P \in C_0^1(D)$) we must have $\int_D h\varphi_P = 0$, so that $h\varphi_P \equiv 0$ in D, contradicting the fact that it is positive on B_ϵ.

Proof of Theorem. By the observation just made, it is enough to show that, for all $\varphi \in C_0^1(D)$:

$$\int_D \Delta(f \circ U)\varphi dV = \int_D [(\Delta f) \circ U]\varphi dV.$$

By multivariable integration by parts, for the left-hand side:

$$\int_D \Delta(f\circ U)\varphi dV = -\int_D \nabla(f\circ U)\cdot\nabla\varphi dV = -\int_D [(\nabla f)U]\cdot\nabla\varphi dV,$$

where we also used the chain rule to assert that $\nabla(f\circ U) = (\nabla f)U$. On the other hand, for the right-hand side:

$$\int_D (\Delta f)\circ U\varphi dV = \int_D [(\Delta f)(\varphi\circ U^{-1})]\circ U dV = \int_D (\Delta f)(\varphi\circ U^{-1})dV,$$

using the change of variables theorem and the facts that D is invariant under U and $|\det U| = 1$.

Again from integration by parts and the chain rule we have:

$$\int_D (\Delta f)(\varphi\circ U^{-1})dV = -\int_D \nabla f\cdot\nabla(\varphi\circ U^{-1})dV = -\int_D \nabla f\cdot[(\nabla\varphi)U^t]dV,$$

where we also use the fact that $U^{-1} = U^t$.

Thus we've reduced the proof to showing that:

$$\int_D [(\nabla f)U]\cdot\nabla\varphi = \int_D \nabla f\cdot[(\nabla\varphi)U^t]dV.$$

But this follows from the (easily verified) fact that for any $n\times n$ matrix A we have:

$$(vA)\cdot w = v\cdot(wA^t), \quad \forall v, w\in\mathbb{R}^n.$$

Application: Laplacian of radial functions. A function $f : B\to R$ (where $B = \{x\in\mathbb{R}^n; |x|\le R\}$ is a ball of some radius) is *radial* if it is invariant under the orthogonal group: $f = f\circ U$, for all $U\in O(n)$. This implies f is a function of distance to the origin only, that is (abusing the notation):

$$f = f(r), \quad \text{if } x = r\omega \text{ with } r = |x|, \omega\in S \text{ (the unit sphere).}$$

The theorem implies the Laplacian of a radial function is also radial:

$$f = f\circ U \quad \forall U \Rightarrow \Delta f = \Delta(f\circ U) = (\Delta f)\circ U \quad \forall U,$$

or $(\Delta f)(r\omega) = g(r)$, for some function $g(r)$ depending on f, which we now compute.

From the divergence theorem on a ball B_R of radius R centered at 0:

$$\int_{B_R} \Delta f dV = \int_{S_R} \frac{\partial f}{\partial n} dS,$$

or:

$$\int_0^R \int_S g(r) r^{n-1} d\omega dr = \int_S f_r(R) R^{n-1} d\omega,$$

and since both integrals in ω just give the (n-1)-dimensional area of S:

$$\int_0^R g(r) r^{n-1} dr = f_r(R) R^{n-1}.$$

Differentiating in R we find:

$$g(R)R^{n-1} = f_{rr}(R)R^{n-1} + (n-1) f_r(R) R^{n-2}$$

(we assume $n \geq 2$), so finally

$$g(r) = \Delta f(r) = f_{rr} + \frac{n-1}{r} f_r.$$

Remark. For general (not necessarily radial) functions we have in polar coordinates $x = r\omega$:

$$\Delta f = f_{rr} + \frac{n-1}{r} f_r + \frac{1}{r^2} \Delta_S f,$$

where Δ_S is the spherical Laplacian:

$$\Delta_S f = f_{\theta\theta} \quad (n=2), \qquad \Delta_S f = f_{\phi\phi} + \frac{\cos\phi}{\sin\phi} f_\phi + \frac{1}{\sin^2\phi} f_{\theta\theta} \quad (n=3),$$

in polar coordinates (r, θ) (resp. spherical coordinates (r, ϕ, θ), $\phi \in [0, \pi], \theta \in [0, 2\pi]$). Note the analogy between Δ_S for n=3 and the Laplacian for $n = 2$:

$$\Delta f = f_{rr} + \frac{1}{r} f_r + \frac{1}{r^2} \Delta_S f \quad (n=2).$$

Making the substitutions:

$$r \to \sin\phi; \quad \frac{1}{r} = (\log r)_r \to \frac{\cos\phi}{\sin\phi} = (\log \sin\phi)_\phi, \quad \Delta_S f \to f_{\theta\theta},$$

we obtain $\Delta_S f$ for n=3.

More generally, the spherical Laplacian Δ_{S^n} on the unite sphere $S^n \subset \mathbb{R}^{n+1}$ can be described inductively in terms of the spherical Laplacian on the unit sphere $S^{n-1} \subset \mathbb{R}^n$. To do this, write $\omega \in S^n$ in spherical coordinates: as a vector in R^{n+1},

$$\omega = (\sin\phi, (cos\phi)\theta) \in \mathbb{R} \times \mathbb{R}^n, \quad \theta \in S^{n-1}, \phi \in [0, \pi].$$

(This represents S^{n-1} as the equator $\phi = 0$ in S^n.) In these coordinates, we have, for a function $f = f(\phi, \theta)$ defined on S^n:

$$\Delta_{S^n} f = f_{\phi\phi} + (n-1)\frac{\cos\phi}{\sin\phi} f_\phi + \frac{1}{\sin^2\phi}\Delta_{S^{n-1}} f.$$

Here the operator $\Delta_{S^{n-1}} f$ involves only differentiation in the variables $\theta \in S^{n-1}$.