Math 231 spring 2009- Exam 3, 4/16/09. No credit for answers without justification. Closed books, closed notes. Calculators are NOT allowed. Time given: 75 minutes. **(Five problems, 20 pts each).**

1. For the initial-value problem:

$$y' = f(t, y), \quad y = y(t), \quad t \in [a, b], \quad y(a) = y_0.$$

(i) Write down the recursion relation that defines the *midpoint Euler* approximation y_n of the solution in $[a, b]$, for a given step size $h = (b - a)/N > 0$.

(ii) Why is *Runge-Kutta 4* called a 'fourth-order method'? (Define briefly the terms that appear in your answer.)

2. A particle of unit mass moves in a plane under an attractive radial force of magnitude $f(r) = \frac{100}{r^2}$ (where r is the distance to the origin). At some moment $t = t_0$ the particle is at an extreme point of its path (that is, $r'(t_0) = 0$), with $r(t_0) = 10$ and speed $v_0 > 0$.

(i) Show that if $v_0^2 > 20$ the path is a hyperbola, if $10 < v_0^2 < 20$ the path is an ellipse and $r(t_0)$ is a minimum of $r(t)$, and if $0 < v_0^2 < 10$ the path is an ellipse and $r(t_0)$ is maximal.

(ii) Find the total energy of the motion, as a function of v_0.

(*Hint:* Recall the speed squared v^2 equals $(r')^2 + r^2(\theta')^2$, and the angular momentum $l = r^2\theta'$ equals $r(t_0)v_0$ at an extreme point.)

3. A particle of unit mass moves on a line, subject to a force:

$$f(y) = 8y - 4y^3.$$

(i) Find the potential $U(y)$ and sketch its graph; find and classify the equilibria (as stable or unstable).

(ii) If $y(0) = 1.5$ and $y'(0) = 0$, find the range of the motion. (That is, find the interval to which the particle's motion is confined.)

4. Solve (explicitly) the initial-value problem:

$$y'' = 2yy', \quad y = y(t), \quad y(0) = 1, y'(0) = 2.$$

Include the domain of the solution in the answer (an interval).

5. (i) Find $Y(s)$, the Laplace transform of the solution $y(t)$ to the initial-value problem: $y'' - 2y' + y = te^t \quad y(0) = y'(0) = 0$. (ii) Find an inverse Laplace transform of the function:

$$Y(s) = \frac{1}{(s - 1)^4}.$$

BRIEF TABLE OF LAPLACE TRANSFORMS.

$f(t)$	$F(s)$
t	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$e^{at}f(t)$	$F(s-a)$
$\theta(t-a)f(t-a)$	$e^{-as}F(s)$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$tf(t)$	$-F'(s)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$