Uniqueness for second-order linear equations.

Consider the initial-value problem for a general second-order linear equation:

$$y'' + p(t)y' + q(t)y = f(t), \quad y = y(t), \quad y(t_0) = y_0, y'(t_0) = y_1.$$

Uniqueness for this problem is the statement that there can be no more than one solution; *existence* is the claim that a solution indeed exists.

The proofs of existence and uniqueness (in the general, variable-coefficient case) are normally not presented in an elementary course. *Existence* for second-order equations with constant coefficients is elementary, so it is part of the course (exponential solutions suffice, as seen in any textbook.) However, even for *constant coefficient* second-order equations, the uniqueness statement (left unproven in all elementary texts I know) underlies claims like 'the general solution has such-and-such form', which *are* part of elementary courses; this is unsatisfying.

Note that the situation is completely different for first-order linear equations, even with variable coefficients, $y' + p(t)y = f(t), y(t_0) = y_0$. Here the difference of two solutions satisfies the corresponding equation with $f \equiv 0$ and $y_0 = 0$, so we have:

$$\frac{d}{dt}(e^{\int_{t_0}^{t} p(s)ds} y(t)) \equiv 0,$$

hence this function is constant and vanishes at $t = t_0$, so $y \equiv 0$. There is no analogous elementary argument for second-order equations, even with *constant* coefficients.

The proof is less elementary in that it involves absolute values and inequalities, but is not hard. We do it for the constant-coefficient case, though the same argument works for variable coefficients. The difference of two solutions of the i.v.p. is a solution $y(t)$ (defined for all $t \in \mathbb{R}$) of the equation with zero initial conditions:

$$y'' + by' + cy = 0, \quad y(0) = 0, y'(0) = 0.$$

We must show this implies $y \equiv 0$. Fix some $T > 0$, and let:

$$M = \max\{|y(t)|; t \in [-T, T]\}.$$

Our goal is to show that $M = 0$. Integrating the equation once we find (using $y'(0) = 0$):

$$y'(t) = -by(t) - c\int_0^t y(s)ds,$$

and integrating again (using $y(0) = 0$):

$$y(t) = -b\int_0^t y(u)du - c\int_0^t (\int_0^u y(s)ds)du.$$

Recalling the basic inequality for integrals (seen in Calculus II):

$$|\int_0^t f(s)ds| \leq \int_0^{|t|} |f(s)|ds,$$

we see that:

$$|y(t)| \leq |b|\int_0^{|t|} |y(u)|du + |c|\int_0^{|t|} (\int_0^u |y(s)|ds)du.$$

Recalling the definition of M, this implies, for $|t| \leq T$:

$$M \leq |b|M|t| + |c|M(t^2/2) \leq (|b|T + |c|(T^2/2))M.$$

Therefore, if we choose T so small that $|b|T + |c|(T^2/2) < 1/2$, we will conclude $M \leq (1/2)M$, which is only possible if $M = 0$. This shows $y(t)$ vanishes in a whole interval (of radius T) centered at 0. Applying this argument to the function $z(t) = y(t + T/2)$ (which solves the same IVP as y, including zero IC at $t = 0$), we find $z \equiv 0$ in $[T, T]$, which means $y \equiv 0$ in $[T/2, 3T/2]$, and therefore in $[-T, 3T/2]$. Continuing to extend the interval in this fashion, we conclude $y \equiv 0$ on the whole real line.