Math 231, Fall 2008 *Exams 3/3B- Solutions.*

1. Assuming that $y(x) = \sum_{n=0}^{\infty} a_n x^n$, we find:

$$\sum_{n=0}^{\infty} 2a_n x^n - 3\sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty}(n+1)n a_{n+1}x^n = 0.$$

$$(n=0) 2a_2 + 2a_0 = 0 \Rightarrow a_2 = -a_0 \quad (n=1) 2a_1 - 3a_1 + 6a_3 + 2a_2 = 0 \Rightarrow a_3 = \frac{1}{6}(a_1 + 2a_0)$$

$$(n=2) - 4a_2 + 6a_3 + 12a_4 = 0 \Rightarrow a_4 = -\frac{1}{2}a_0 - \frac{1}{12}a_1.$$

The general solution is:

$$y(x) = a_0(1 - x^2 + \frac{1}{3}x^3 - \frac{1}{2}x^4 + O(x^5)) + a_1(x + \frac{1}{6}x^3 - \frac{1}{12}x^4 + O(x^5)).$$

Writing the equation in 'standard form':

$$y'' - \frac{3x}{x+1}y' + \frac{2}{x+1}y = 0,$$

we see that the interval of convergence is $-1 < x < 1$ (the largest interval centered at zero and not containing -1, since on this interval all coefficient functions are analytic).

2.(i) (The question in exam 3 had a (+) sign, in 3B a (-)sign.) The Laplace-transformed equation is:

$$(s^2 - 2s \pm 5)Y(s) = 1 + G(s).$$

With the (+) sign:

$$\frac{1}{(s-1)^2+4} \longrightarrow (1/2)e^t \sin(2t) := f(t);$$

With the (-) sign:

$$\frac{1}{(s-1)^2-6} \longrightarrow (1/2\sqrt{6})e^t(e^{\sqrt{6}t} - e^{-\sqrt{6}t}) := f(t)$$

In either case, the solution is:

$$y(t) = f(t) + (f * g)(t).$$

(ii)$g(t) = 17[u(t-1) - u(t-2)]$ transforms to $G(s) = (17/s)(e^{-s} - e^{-2s})$. Using partial fraction expansions:

$$\frac{1}{s(s^2-2s+5)} = \frac{1}{5}(\frac{1}{s} - \frac{s-2}{(s-1)^2+4}) \longrightarrow \frac{1}{5}(1 - e^t\cos(2t) - \frac{e^t}{2}\sin(2t)) := h(t);$$

with the (-)sign:

$$\frac{1}{s(s^2-2s-5)} = -\frac{1}{5}(\frac{1}{s} - \frac{s-2}{(s-1)^2-6}) \longrightarrow -\frac{1}{5}(1 - e^t\cosh(\sqrt{6}t) - \frac{e^t}{\sqrt{6}}\sinh(\sqrt{6}t)) := h(t).$$

In either case, the solution is:

$$y(t) = f(t) + 17[u(t-1)h(t-1) - u(t-2)h(t-2)] = \begin{cases} f(t), \quad 0 \le t < 1 \\ f(t) + 17h(t-1), \quad 1 < t < 2 \\ f(t) + 17h(t-1) - 17h(t-2), \quad t > 2 \end{cases}$$

To check continuity, we verify that the one-sided limits coincide. At $t = 1$:

$$\lim_{t \to 1^-} h(t-1) = \lim_{t \to 1^+} h(t-1) = \lim_{t \to 0} h(t) = 0,$$

since $h(t)$ is continuous at 0, with value 0. Since $f(t)$ is continuous everywhere, this implies:

$$\lim_{t \to 1^-} y(t) = \lim_{t \to 1^+} y(t) = f(1),$$

so $y(t)$ is continuous at 1 (if we define $y(1) = f(1)$). The same argument shows $y(t)$ is continuous at 2 (if we define $y(2) = f(2)$).

Remark: It is a *general fact* that solutions of linear differential equations with a piecewise continuous 'forcing term' (that is, with only 'jump discontinuities') are continuous, as in this example. However, on a test question this would have to be verified explicitly, as above.

3. Following the hint, we find:

$$v\frac{dv}{dy} = \frac{3v^2}{y+1}, \quad \frac{dv}{v} = 3\frac{dy}{y+1} \quad y' = v = C(y+1)^3,$$

and integrating:

$$\frac{y'}{(y+1)^3} = C, \quad -\frac{1}{(y+1)^2} = 2Cx + D.$$

4.(Exam 3) $r' = e^\theta \theta' = r\frac{l}{r^2} = \frac{l}{r}$, so $r'' = -\frac{l}{r^2}r' = -\frac{l}{r^3}$, and then $f(r) = r'' - \frac{l^2}{r^3} = -\frac{2l}{r^3}$.

(Exam 3B) (i) $U(y) = -\int f(y)dy = -4y^2 + y^4$. The equilibria (critical points of U, or zeroes of f) are $\bar{y} = \pm\sqrt{2}$ (local minima of U, hence stable) and $\bar{y} = 0$ (local max of U, hence unstable.)

(ii) The total energy is $E(y, y') = \frac{(y')^2}{2} - 4y^2 + y^4$, so $E_0 = (1.5)^4 - 4(1.5)^2$. Solving $U(y) = E_0$ we find two positive roots: $y = 1.5$ (of course) and $\sqrt{4 - (1.5)^2} \sim 1.32$, so the range of motion is $[1.32, 1.5]$. (There are also negative solutions, corresponding to the interval $[-1.5, -1.32]$, but this is in a separate 'trough' of the potential, hence is not occupied by the particle.)