GRAPHICAL ANALYSIS OF AUTONOMOUS EQUATIONS

The method described here applies to autonomous first-order equations:

$$y' = f(y), \quad y = y(t).$$

We'll assume f and $\frac{df}{dy}$ are continuous for all y (in fact, they'll usually be polynomials). By the existence-uniqueness theorem, given any $y_0 \in \mathbb{R}$ there is a unique solution to the DE satisfying $y(0) = y_0$. Since autonomous equations are separable, they can in principle be solved by integration; in practice this can be a lot of work, or impossible. The point of the graphical analysis is that many conclusions of practical value- long-time behavior or solutions, whether solutions are defined for all t or not- can be decided without solving the equation.

Example 1. *(Logistic growth.)* Consider

$$y' = ay - by^2, \quad y = y(t),$$

where a, b are positive constants. This is sometimes used as a population growth model, with a representing the intrinsic growth rate and $-by^2$ a negative contribution to growth (say, due to less than friendly encounters between individuals).

To solve the equation, one needs the 'partial fraction expansion':

$$\frac{1}{ay - by^2} = \frac{1}{a}(\frac{1}{y} + \frac{b}{a - by}),$$

and then integration gives:

$$\ln \frac{y}{a - by} = at + C, \text{ or } y(t) = \frac{Cae^{at}}{1 + bCe^{at}},$$

where the value of C can be computed from the initial condition $y(0) = y_0$:

$$C = \frac{y_0}{a - y_0 b}.$$

We may also write the general solution in the form:

$$y(t) = \frac{ae^{a(t-t_0)}}{1 + be^{a(t-t_0)}} \qquad \text{if } C = e^{-t_0} > 0;$$

$$y(t) = -\frac{ae^{a(t-t_0)}}{1 - be^{a(t-t_0)}} \qquad \text{if } C = -e^{-t_0} < 0.$$

($C > 0$ corresponds to $0 < y_0 < a/b$, while $C < 0$ corresponds to $y_0 > a/b$ or $y_0 < 0$).This shows all solution curves can be seen as translations of a small number of curves.

Using y_0 as a parameter, the general solution is:

$$y(t) = \frac{y_0 e^{at}}{1 + y_0(b/a)(e^{at} - 1)}.$$

The denominator vanishes when $e^{at} - 1 = -\frac{a}{by_0}$. From this we see the following:

(i) If $y_0 < 0$, the solution exists only in an interval of the form $(-\infty, T_*)$, where $T_* = (1/a)\ln(1 - \frac{a}{by_0}) > 0$; $y(t) \to -\infty$ as $t \to T_*$ from the left.

(ii) If $y_0 > 0$, there are two cases: for $0 < y_0 < a/b$, we have that $-\frac{a}{by_0} < -1$, while $e^{at} - 1 > -1$ for all t. So in this case the denominator does not vanish, and the solution is defined for all $t \in \mathbb{R}$.

If, however, $y_0 > a/b$ there will be a $T_* < 0$ so that the denominator vanishes at $t = T_*$. So in this case the solution is defined only on the interval (T_*, ∞) (where T_* depends on y_0), and tends to $+\infty$ as $t \to T_*$ from the right.

In general, if $f(y)$ has 'growth rate greater than linear' in y as $|y| \to \infty$, a solution will 'blow up' at some finite positive or negative time, unless it is 'trapped' between two constant solutions.

As noted above, within each range (defined by the equilibria) all solution curves have the same 'shape', and differ only by translation. This is a general fact for autonomous equations.

Note that $y(t) \equiv a/b$ is itself a *constant solution* to the equation, which is *missed* by the 'general solution' found (that is, it does not correspond to any value of C).

A constant solution such as a/b in this example is called 'stable'. Precisely, a constant solution $\bar{y}$ to an autonomous DE is said to be *stable* if any solution $y(t)$ with initial condition $y(0) = y_0$ *sufficiently close* to $\bar{y}$ tends to $\bar{y}$ as $t \to \infty$. Constants solutions are also called 'equilibria' (from mechanics: they correspond to positions from which particles following the equation don't move.) An equilibrium $\bar{y}$ is *unstable* if solutions starting from y_0 sufficiently close to $\bar{y}$ move away from $\bar{y}$ for positive time (equivalently: approach $\bar{y}$ as $t \to -\infty$.

In the next example we find and classify the equilibria without solving the equation.

Example 2. Consider the autonomous equation:

$$y' = f(y), \quad y = y(t), \quad f(y) = y(y-2)(y+1).$$

The equilibria (constant solutions) correspond exactly to the values of y for which $f(y) = 0$, in this case: $\bar{y} = 0, -1$ or 2. To decide their stability, we look at the sign of f in each of the four intervals defined by the equilibria. From left to right, these are: $- \quad + \quad - \quad +$. In terms of the mechanical analogy (where y' represents velocity, $f(y) > 0$ corresponds to 'particle moving to the right', and $f(y) < 0$ to 'particle moving to the left', so we may draw a diagram with arrows representing the direction of motion:

$$\longleftarrow -1 \longrightarrow 0 \longleftarrow 2 \longrightarrow$$

From this we see immediately that -1 and 2 are unstable equilibria, while 0 is stable.

Next we draw the diagram of all solutions (*done in class.*) To begin, in the (t, y) plane we have horizontal lines at $y = -1, 0, 2$, corresponding to constant solutions. Solutions with $-1 < y_0 < 0$ move away from the line $y = -1$, towards the line $y = 0$, as $t \to \infty$; solutions with $0 < y_0 < 2$ move away from the line $y = 2$, towards the line $y = 0$. These solutions are all defined for all $t \in \mathbb{R}$ (they are 'trapped between constant solution', so cannot 'blow up in finite time'.) On the other hand, if $y_0 > 2$ or $y_0 < -1$, the situation is different: this solutions move away from $y = 2$ (or $y = -1$, respectively) for positive time. In fact, for large y we have $f(y) \sim y^3$, and we know the solutions of $y' = y^3$ exhibit 'finite-time blowup'. From this it is not hard to show that solutions with $y_0 > 2$ or $y_0 < -1$ are only defined in an interval $(-\infty, T_*)$, where the 'blowup time' T_* depends on y_0.

The following examples may be analyzed in the same way. In each case, we (i) find and classify the equilibria (as stable or unstable); (ii) sketch the diagram of all solutions (including at least two curves in the region defined by the equilibria); (iii) identify the initial conditions corresponding to solutions defined for all $t \in \mathbb{R}$, and those defined only on an interval.

1.$y' = -2y(y^2 - 1)$
2.$y' = \frac{\sin y}{1+y^2}$
3.$y' = 2y^2 - 4y$
4.$y' = e^{-y} \cos y$
5.$y' = (y^2 + 1)(y^2 - 1)$