Autonomous Riccati equations

By direct computation, we find that $y(t) = \alpha \tanh(\alpha t)$ and $y(t) = \alpha \coth(\alpha t)$ are both solutions of the non-linear first-order DE:

$$y' + y^2 - \alpha^2 = 0;$$

the first one defined for all $t \in \mathbb{R}$, the second in either interval $(0, \infty)$ or $(-\infty, 0)$ (the domains of solutions to ODE are always taken to be intervals). This equation also has the constant solutions $y(t) \equiv \alpha$ and $y(t) \equiv -\alpha$ (check that they are solutions). A similar computation for $y(t) = \alpha \tan(\alpha t)$ and $y(t) = \alpha \cot(\alpha t)$ leads to the two DE (respectively):

$$y' - y^2 - \alpha^2 = 0, \quad y' + y^2 + \alpha^2 = 0.$$

Unlike the first one, these last two equations *do not* have constant solutions (check this!) These three equations are all examples of *autonomous* first-order differential equations; '*autonomous*' means the independent variable (t in this case) does not occur explicitly in the equation.

A very general expression for a first-order (nonlinear) ODE is:

$$y' = f(t, y), \quad y = y(t),$$

where f is an arbitrary function of two variables. An autonomous equation is of the special form:

$$y' = f(y).$$

The 'physical interpretation' (of 'autonomous') is: the rate of change of the system being analyzed depends on the *state* of the system at time t (that is, on the value of y at time t), but does not depend on time (except to the extent that y changes with t.) For example, the general linear first-order equation with constant coefficients:

$$y' = ky + r, \quad k, r \in \mathbb{R} \quad \text{(constants)}$$

is autonomous, but if the coefficients depend on t:

$$y' = k(t)y + r(t)$$

is is not. Autonomous equations have the following property: *if $y(t)$ is a solution, then for any constant $c \in \mathbb{R}$, the function $y_c(t) = y(t + c)$ is also a solution:*

$$y_c'(t) = y'(t + c) = f(y(t + c)) = f(y_c)(t).$$

For example, the function $\cosh(3t-2)$ solves the same autonomous DE as $\cosh(3t)$ (with different domains: the second is defined for $t > 0$ (or $t < 0$), the first for $t > 2/3$ (or $t < 2/3$)).

In class we saw that if we look at all the translations of graphs of solutions of $y' + y^2 - 1 = 0$ (i.e., the graphs of the solutions $\cosh(t+c)$, $\sinh(t+c)$ for all $c \in \mathbb{R}$, plus the constant solutions $y(t) \equiv 1$ and $y(t) \equiv -1$), these graphs *fill up the whole* (t, y) *plane without intersecting each other*. The same is true if we look at all the translations of the graphs of $\tan(t)$, or of the graphs of $\cot(t)$. This is related to the main theorem in this course, the *existence-uniqueness theorem for first-order ODE*, which informally stated says that, for autonomous equations $y' = f(y)$ (assuming the first derivative df/dy is a continuous function):

For any given (t_0, y_0)*, there is exactly one solution* $y = y(t)$ *of the equation* $y' = f(y)$*, defined in some open interval containing* t_0*, whose graph goes through the point* (t_0, y_0) (that is, $y(t_0) = y_0$.)

If you think about this a little, it says that through every point in the (t, y) plane we'll find exactly one 'solution curve' (=graph of a solution), which really amounts to the 'filling' property verified in the examples above by direct observation. (*Remark*: the E/U theorem is also true for the more general non-autonomous equations)

It is important to note that the largest interval where a solution is defined is rarely all of $\mathbb{R}$ (unless the equation is linear, with coefficients $k(t)$, $r(t)$ continuous on $\mathbb{R}$) Consider the following examples (for complete understanding, plot the graph of each function):

Ex.1 The solution of $y' + y^2 - 4 = 0$ with $y(-1) = -2\coth(5)$ is $y(t) = 2\coth(2t-3)$, defined on the interval $(-\infty, 3/2)$. (The other possible choice is the interval $(3/2, \infty)$, but this does not include the point $t_0 = -1$ where the value of the solution is prescribed.)

Ex.2 The solution of $y' - y^2 - 9 = 0$ with $y(\pi) = 0$ is $y(t) = 3\tan(3t)$, defined on the interval $(\pi - \pi/6, \pi + \pi/6)$ (ignore the other intervals where $\tan(3t)$ is defined).

Ex.3 The solution of $y' + y^2 - 4 = 0$ with $y(-1) = 2\tanh(3)$ is $y(t) = 2\tanh(2t+5)$, defined on all of $\mathbb{R}$.

A consequence of this discussion is that the general (non-constant) solutions of each equation described above are obtained from the basic solutions by translation. For example, the general non-constant solution of:

$$y' + y^2 - 4 = 0$$

is $y(t) = 2\tanh(2t + C)$ if $|y(t_0)| < 2$, $y(t) = 2\coth(2t + C)$ if $|y(t_0)| > 2$.

Exercises.

1. Suppose $f = f(t)$ is a solution of the autonomous equation:

$$f' + f^2 + k = 0, \quad k \in \mathbb{R}.$$

What is the autonomous equation solved by $g(t) = -f(t)$?

2. Solve the initial-value problems below for $y = y(t)$, including the largest interval where the solution is defined. Sketch the graph of the solution.

$(i) y' + y^2 - 25 = 0, \quad y(0) = 3.$
$(ii) y' - y^2 + 16 = 0, \quad y(-1) = 5.$
$(iii) y' - y^2 - 9 = 0, \quad y(1) = 1.$
$(iv) y' + y^2 + 16 = 0, \quad y(-1) = -1.$