

MATH 142, Fall 2006- Some interesting integrals of algebraic functions

1. Prove the reduction formula: if $I_n = \int dx/(x^2 + a^2)^n$,

$$I_n = \frac{x}{(2n-2)a^2(x^2 + a^2)^{n-1}} + \frac{2n-3}{(2n-2)a^2} I_{n-1}.$$

Hint:

$$\frac{1}{(x^2 + a^2)^{n-1}} = x \frac{x}{(x^2 + a^2)^n} + a^2 \frac{1}{(x^2 + a^2)^n},$$

use integration by parts for the first term (splitting it as indicated)

2. $\int \frac{x^2 dx}{(a^2 - x^2)^{3/2}}$ ($x = a \sin t$)

$$Ans. \quad \frac{x}{\sqrt{a^2 - x^2}} - \arcsin\left(\frac{x}{a}\right) + C.$$

3. $\int \frac{\sqrt{2x+x^2}}{x^2} dx$

Hint: completing squares, we set $u = x + 1$, then $u = 1/\cos t$. This leads to the integrals of $\frac{1}{\cos t}$ and of $\frac{2}{(1-\cos t)} = \frac{1}{\sin^2(t/2)}$, both 'standard': the second gives $-2\cotan(t/2)$, while the first equals:

$$\int \frac{\cos t}{1 - \sin^2 t} dt = \int \frac{dw}{1 - w^2} = \frac{1}{2} \ln \frac{1+w}{1-w} = \ln\left(\frac{1+\sin t}{\cos t}\right).$$

$$Ans. \quad -2 \frac{\sqrt{2x+x^2}}{x} + \ln(x+1 + \sqrt{2x+x^2}) + C.$$

4. $\int \frac{x^2 dx}{\sqrt{1-2x-x^2}}$. (complete squares)

$$Ans. \quad \frac{1}{2}(3-x)\sqrt{1-2x-x^2} + 2 \arcsin\left(\frac{x+1}{\sqrt{2}}\right) + C.$$

5. $\int x\sqrt{x+ax} dx$ ($u^2 = x+a$).

$$Ans. \quad -\frac{2}{15}(a+x)^{3/2}(2a-3x) + C$$

6. $\int \frac{x}{\sqrt{x+1+(x+1)^{1/3}}} dx$ ($u^6 = x+1$)

$$Ans \quad \frac{2}{3}(x+1)^{3/2} - \frac{3}{4}(x+1)^{4/3} + \frac{6}{7}(x+1)^{7/6} - x - 1 + \frac{6}{5}(x+1)^{5/6} - \frac{3}{2}(x+1)^{2/3}.$$

$$7. \int \frac{x^2 dx}{(1+x)^{1/3}} \cdot (u^3 = x + 1)$$

$$Ans. \quad \frac{3}{40}(x+1)^{2/3}(9-6x+5x^2)$$

$$8. \int \frac{dx}{x^{2/3}-x^{1/2}} \quad (u^6 = x)$$

$$Ans. \quad 3x^{1/3} + 6x^{1/6} + \ln|x^{1/6} - 1| + C.$$

$$9. \int \frac{dx}{\sqrt{x(x-1)}} \quad (u^2 = x)$$

$$Ans. \quad \ln \left| \frac{\sqrt{x}-1}{\sqrt{x}+1} \right| + C$$

$$10. \int \sqrt{a^2 - x^2} dx. \quad (x = a \sin t)$$

$$Ans. (1/2)[x\sqrt{a^2 - x^2} + a^2 \arcsin(x/a)] + C.$$

Source: Calculus of One Variable, by Joseph W. Kitchen- Addison-Wesley, 1968 (the textbook I used when I learned calculus. Some of the answers obtained from Maple 10.)