CONIC SECTIONS

1. Geometric definition. Ellipses, hyperbolas and parabolas have geometric definitions as *loci* of points in the plane with certain properties. Fix a line l and a point F not in l.

The *parabola* with focus F and directrix l is the locus of points P in the plane whose distances to F and l are equal:

$$|PF| = \text{dist}(P, l).$$

A parabola has an axis of symmetry- the perpendicular from F to l- and a vertex, the point on this axis halfway between F and l. Only one geometric parameter affects the shape of a parabola: the distance d from F to l.

Fix a number $0 < e < 1$. The *ellipse* with focus F, directrix l and 'eccentricity' e is the locus of points P in the plane satisfying:

$$|PF| = e\text{dist}(P, l).$$

Again the perpendicular to l through F is an axis of symmetry. There is a second focus $\bar{F}$ on the axis, and a corresponding directrix $\bar{l}$ parallel to l, so the ellipse can be described as the same locus for $\bar{F}$ and $\bar{l}$. The median perpendicular of the segment $\bar{F}F$ is a second axis of symmetry. The shape depends on *two* geometric parameters: e and d, the distance from F to l.

2. Equations. In a cartesian coordinate system with the origin at the vertex of the parabola, the x-axis parallel to the directrix and the y-axis along the parabolic axis, the equation of the parabola is very simple:

$$y = \frac{1}{2d}x^2$$

(F is the point $(0, d/2)$ and l is the line $y = -d/2$). In polar coordinates with the origin at the focus F and the radial direction $\theta = 0$ given by the axis parabola from F to the vertex, from the geometric definition one directly obtains the equation:

$$r = \frac{d}{1 + \cos\theta}, \quad \theta \in (-\pi, \pi).$$

For the ellipse, the same choice of polar coordinates and the geometric definition lead just as directly to the polar equation:

$$r = \frac{ed}{1 + e\cos\theta}, \quad \theta \in \mathbb{R}.$$

(This is periodic in θ, hence defines a closed curve.) For cartesian coordinates, choose as the x-axis the axis of the ellipse, and as y-axis the median perpendicular of the focal segment. The *scale* is fixed by letting the focus F have coordinates $(c, 0)$, with the 'inspired choice':

$$c = \frac{e^2 d}{1 - e^2}.$$

Then the geometric definition leads to the cartesian equation:

$$(1 - e)^2 x^2 + y^2 = \frac{e^2 d^2}{1 - e^2},$$

which we can identify with the standard form:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

by defining:

$$a = \frac{ed}{1 - e^2}, \quad b = \frac{ed}{\sqrt{1 - e^2}}.$$

With these choices, we can confirm the well-known relations (*exercise*):

$$e = \frac{c}{a}, \quad a^2 = b^2 + c^2.$$

In addition, applying the defining property of the ellipse to the vertex $(a, 0)$, we find that $a - c = e(d - a + c)$, which leads to the useful relation (*exercise*):

$$d = \frac{a}{e}(1 - e^2) = \frac{b^2}{c}.$$

Another way to state this is by saying the directrices have equations $x = \pm(c/e^2) = \pm(a/e)$. Denoting by V and $\bar{V}$ the vertices on the line through the foci, this means:

$$\frac{|\bar{V}V|}{\mathrm{dist}(l, \bar{l})} = \frac{|\bar{F}F|}{|\bar{V}V|} = e,$$

3. Properties. The conics have very interesting *reflection properties.* These follow directly from the geometric definitions and the following easily demonstrated 'fact from mechanics':

Fact. Let $\mathbf{r}(t)$ be a parametrized curve (in the plane or in space). For a given t, let $\theta(t)$ be the angle formed by $\mathbf{r}(t)$ and the velocity vector $\mathbf{r}'(t)$. Then the instantaneous rate of change of distance to the origin is:

$$\frac{dr}{dt} = ||\mathbf{r'}|| \cos\theta.$$

Reflection property of the parabola: At a point P of a parabola, the tangent to the parabola makes equal angles with the line from P parallel to the axis and the line from P to the focus F.

Proof. Denote by α the first angle, by β the second. Let $\mathbf{r}(s) = (x(s), y(s))$ be the arc-length (speed one) parametrization of the parabola, with the origin at the focus. From the geometry, we see that the angle between position and velocity vectors at P is also β, and also that the velocity vector is $(x'(s), y'(s)) = (-\cos\alpha, \sin\alpha)$. The defining property of the parabola can be written as:

$$r(s) = d - x(s).$$

Differentiating this relation and using the fact above, we obtain:

$$\cos\beta = r'(s) = -x'(s) = \cos\alpha.$$

Since both angles are taken in $(0, \pi)$, this shows $\alpha = \beta$.

Before introducing the reflection property for the ellipse, recall the 'string property': the sum of the lengths of the segments drawn from a point P on the ellipse to the foci F, $\bar{F}$ is constant throughout the ellipse. (This makes it possible to draw a pretty good ellipse with a string tied to two pins at the foci.) This is easy to see if we apply the defining property to both focus-directrix pairs (F, l) and $(\bar{F}, \bar{l})$:

$$|PF| + |P\bar{F}| = e \text{ dist}(P, l) + e \text{ dist}(P, \bar{l}) = e \text{ dist}(l, \bar{l}),$$

which is independent of P.

Reflection property of the ellipse: At any point P of an ellipse, the lines drawn from P to the foci F and $\bar{F}$ make equal angles with the tangent to the ellipse at P.

Proof. Denote by $\mathbf{r}(s)$ and $\bar{\mathbf{r}}(s)$ arc-length parametrizations of the ellipse, with the origin taken at F and at $\bar{F}$, respectively. Denote by α (resp. $\bar{\alpha}$) the angle made by the tangent at P and the segment from P to F (resp. $\bar{F}$.) With the parametrizations running clockwise (and $\bar{F}$ to the left of F), we see that the angle between position and velocity at P for $\bar{\mathbf{r}}(s)$ is $\bar{\alpha}$, and for $\mathbf{r}(s)$ is $\pi - \alpha$. Differentiating the 'string property' $r(s) + \bar{r}(s) = const$ and using the 'fact from mechanics', we obtain:

$$\cos\bar{\alpha} + \cos(\pi - \alpha) = \bar{r}'(s) + r'(s) = 0,$$

and it follows that $\cos\alpha = \cos\bar{\alpha}$, and hence $\alpha = \bar{\alpha}$.

The *hyperbolas* have analogous properties, and the proofs will be left as *exercises* for the reader. The focus-directrix definition is the same as for the ellipse: a hyperbola is the locus of points in the plane whose distances to a fixed point F and to a fixed line l are in a constant ratio:

$$|FP| = e \text{ dist } (P, l),$$

where in contrast to the ellipse we now take $e > 1$.

Exercise 1. From this definition, derive the equation in polar coordinates:

$$r = \frac{de}{1 + e\cos\theta}, \quad -\theta_0 < \theta < \theta_0,$$

where $d = \text{dist}(F, l)$ and $\theta_0 = \arccos(-1/e) \in (0, \pi)$. This represents only one branch of the hyperbola. What is the polar representation of the other branch?

Like the ellipse, the hyperbola has two foci F and $\bar{F}$, and an axis of symmetry going through them. The median perpendicular of the foci is a second axis of symmetry, parallel to the two directrices l and $\bar{l}$. To establish the cartesian equation, let the x and y axes be these axes of symmetry, and choose the scale so that the foci have coordinates $(\pm c, 0)$, where $c = e^2 d/(e^2 - 1)$.

Exercise 2.(i) Derive the cartesian equation:

$$(e^2 - 1)x^2 - y^2 = \frac{e^2 d^2}{e^2 - 1}.$$

(ii) Verify that this is equivalent to the standard form:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

if we set:

$$a = \frac{ed}{e^2 - 1}, \quad b = \frac{ed}{\sqrt{e^2 - 1}}.$$

From this the well-known relations $c^2 = a^2 + b^2$ and $e = c/a$ follow easily.

(iii) Show that $d = \frac{e^2-1}{e^2} c$, and conclude that the directrices are the lines $x = \pm(c/e^2) = \pm(a/e)$. If $\bar{V}$ and V are the vertices, we can express this in the form:

$$\frac{|\bar{V}V|}{\text{dist}(l, \bar{l})} = \frac{|\bar{F}F|}{|\bar{V}V|} = e,$$

just as for the ellipse (but now $e > 1$).

Exercise 3- pseudostring property. The *difference* between the distances from a point on the curve to the two foci is constant on each branch of the hyperbola (a positive constant on one branch, a negative constant on the other):

$$|PF| - |P\bar{F}| = const.$$

It is part of the exercise to propose a better name for this property.

Exercise 4-reflection property. Show that the angle $\bar{F}PF$ formed by the segments drawn from a point P on the hyperbola to the two foci is bisected by the tangent at P.

4. Arc length.

Parabolas. For the parabola $y = \frac{1}{2d}x^2$, the arc length from $x = 0$ to $x = A$ is given by:

$$\int_0^A \sqrt{1 + x^2/d^2}dx = d\int_0^{A/d} \sqrt{1+u^2}du.$$

The integral is computable (say, by trigonometric substitution), and yields:

$$L_{parabola}(x = 0, x = A) = d[\frac{a}{2}\sqrt{1+a^2} + \ln(a + \sqrt{1+a^2})],$$

where $a = A/d$.

Ellipses. From the cartesian equation we obtain y as a function of x:

$$y(x) = \frac{b}{a}\sqrt{a^2 - x^2}, \quad 0 \le x \le a$$

and the element of arc length:

$$ds = \sqrt{\frac{a^4 - c^2x^2}{a^4 - a^2x^2}}dx.$$

With the substitution $t = x/a$, this can be written as:

$$ds = a\sqrt{\frac{1 - e^2t^2}{1 - t^2}}dt,$$

where $e = \frac{c}{a} \in (0, 1)$ is the eccentricity. Thus the length of the arc from $x = 0$ to $x = A$ (where $A \le a$) is given in terms of *Legendre's elliptic integral of the second kind:*

$$\mathbb{E}(X, e) := \int_0^X \sqrt{\frac{1 - e^2x^2}{1 - x^2}}dx, \quad X \in [0, 1], \quad 0 < e < 1$$

by the simple formula:

$$L_{ellipse}(x=0, x=A) = a\mathbb{E}(\frac{A}{a}, e).$$

The function $\mathbb{E}(X, e)$ is standard in mathematical software, as is *Legendre's elliptic integral of the first kind:*

$$\mathbb{F}(V, f) := \int_0^V \frac{1}{\sqrt{1+f^2v^2}\sqrt{1+v^2}}dv, \quad V > 0, \quad f \in \mathbb{R}.$$

Hyperbolas. From the cartesian equation, expressing x as a function of y:

$$x(y) = \frac{a}{b}\sqrt{b^2+y^2}, \quad y \geq 0$$

we obtain (using $c^2 = a^2 + b^2$):

$$ds = \sqrt{1+(x')^2(y)}dy = \frac{\sqrt{c^2y^2+b^4}}{\sqrt{b^2y^2+b^4}}dy.$$

Now make the substitution (not the first one one would think of) $v = \frac{c}{b^2}y$ to get:

$$ds = cf^2\sqrt{\frac{1+v^2}{1+f^2v^2}}dv, \quad f^2 = \frac{b^2}{c^2} < 1.$$

We see that finding the length of a hyperbolic arc comes down to computing the elliptic integral:

$$\mathbb{L}(V, f) = \int_0^V \sqrt{\frac{1+v^2}{1+f^2v^2}}dv,$$

and for that we need to express it in terms of the more standard ones $\mathbb{E}$ and $\mathbb{F}$. I'm sorry to report that this is very tricky (the problem was solved 200 years ago, possibly first by Legendre.) First note that the change of variable $x = v/\sqrt{1+v^2}$ allows one to express the elliptic integral of second kind in the form:

$$\mathbb{E}(X, e) = \int_0^V \frac{\sqrt{1+f^2v^2}}{(1+v^2)^{3/2}}dv, \quad f^2 = 1-e^2, X = V/\sqrt{1+V^2}.$$

Now there are two main steps; the first is an exercise in differentiation, and in the second step we rearrange the terms so they look like the integrands we are interested in:

$$[v\sqrt{\frac{1+f^2v^2}{1+v^2}}]' = \sqrt{\frac{1+f^2v^2}{1+v^2}} - \frac{(1-f^2)v^2}{\sqrt{1+f^2v^2}(1+v^2)^{3/2}}$$

$$= f^2\sqrt{\frac{1+v^2}{1+f^2v^2}} - \frac{f^2}{\sqrt{1+f^2v^2}\sqrt{1+v^2}} + \frac{\sqrt{1+f^2v^2}}{(1+v^2)^{3/2}}.$$

(*Exercise:* check the algebra.)

This immediately implies the integral $\mathbb{L}$ admits the expression in terms of $\mathbb{E}$, $\mathbb{F}$ and an algebraic function:

$$f^2\mathbb{L}(V,f) = f^2\mathbb{F}(V,f) - \mathbb{E}(X,e) + V\sqrt{\frac{1+f^2V^2}{1+V^2}},$$

where $e = \sqrt{1-f^2}$, $X = V/\sqrt{1+V^2}$. This shows the length of a hyperbolic arc can be computed using elliptic integrals of first and second kinds. The length of the arc from $y=0$ to $y=B$ is:

$$L_{hyperbola}(y=0, y=B) = cf^2\mathbb{L}(\frac{cB}{b^2}, f).$$