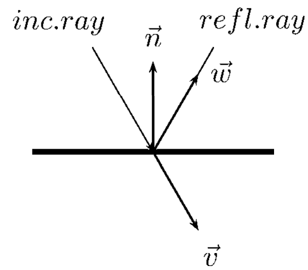
REFLECTION AND REFRACTION ON GENERAL PLANES

Let $\vec{v}$ be the unit vector in the direction of the incident ray. Consider reflection on a plane with unit normal $\vec{n}$ (pointing *into* the half-space containing the light source). Experimentally, we know that (i) the incident ray and the reflected ray make the same angles with the normal; (ii) incident ray, reflected ray and the normal are in the same plane. The **problem** is to find the direction $\vec{w}$ of the reflected ray (also given by a unit vector).

Remark: Since we're considering vectors, an origin has implicitly been chosen; we assume the origin is the point in the plane where reflection takes place.

Notation: We use $\langle \vec{a}, \vec{b} \rangle$ for the scalar product of two vectors $\vec{a}, \vec{b}$.



SOLUTION: From condition (ii), we may express $\vec{w}$ as a linear combination of $\vec{v}$ and $\vec{n}$:

$$\vec{w} = c\vec{v} + d\vec{n}.$$

Condition (i) implies $\langle \vec{w}, \vec{n} \rangle = -\langle \vec{v}, \vec{n} \rangle$. Since:

$$\langle \vec{w}, \vec{n} \rangle = c\langle \vec{v}, \vec{n} \rangle + d,$$

this implies:

$$(c+1)\langle \vec{v}, \vec{n} \rangle + d = 0, \text{ or } d = -(c+1)\langle \vec{v}, \vec{n} \rangle;$$

thus we have the expression for $\vec{w}$:

$$\vec{w} = c\vec{v} - (c+1)\langle \vec{v}, \vec{n} \rangle \vec{n}.$$

Now substitute this in the equation $||\vec{w}||^2 = \langle \vec{w}, \vec{w} \rangle = 1$ and expand to get:

$$c^2 - 2c(c+1)\langle \vec{v}, \vec{n} \rangle^2 + (c+1)^2\langle \vec{v}, \vec{n} \rangle^2 = 1,$$

or, after cancellations:

$$c^2 - c^2\langle \vec{v}, \vec{n}\rangle^2 + \langle \vec{v}, \vec{n}\rangle^2 = 1.$$

Equivalently, $(1-c^2)(1-\langle \vec{v}, \vec{n}\rangle^2) = 0$, and we conclude $c = \pm 1$ (unless $\vec{v} = -\vec{n}$, in which case we know anyway that $\vec{w} = \vec{n}$). Since the components of $\vec{v}$ and $\vec{w}$ *perpendicular* to $\vec{n}$ have the same direction, we know the sign is $+$, so $c = 1$ and we have the *ANSWER:*

$$\vec{w} = \vec{v} - 2\langle \vec{v}, \vec{n}\rangle \vec{n}.$$

Exercise: Now consider *refraction* on a plane with unit normal $\vec{n}$, oriented as before. Condition (ii) still holds for the incident and refracted rays, and *Snell's law* (1621) says the angles of incidence and of refraction θ_i, θ_r satisfy:

$$\frac{\sin\theta_i}{\sin\theta_r} = \lambda,$$

for a constant $\lambda > 0$ independent of the direction of the incident ray (the *index of refraction*, equal to the ratio of the speeds of light in the two media involved.) Find an expression for the direction vector $\vec{w}$ of the refracted ray, in terms of the incident direction $\vec{v}$, the normal $\vec{n}$ and λ.

Hint: Here $\cos\theta_i = -\langle \vec{v}, \vec{n}\rangle$ and $\cos\theta_r = -\langle \vec{w}, \vec{n}\rangle$, so:

$$\sin\theta_i = \sqrt{1-\langle \vec{v}, \vec{n}\rangle^2}, \quad \sin\theta_r = \sqrt{1-\langle \vec{w}, \vec{n}\rangle^2}.$$

Start again by writing $\vec{w}$ as a linear combination of $\vec{v}$ and $\vec{n}$, then use Snell's law and the fact that $\vec{w}, \vec{v}, \vec{n}$ are all unit vectors.
ANSWER: $\vec{w} = \frac{1}{\lambda}[\vec{v} - (x - \sqrt{\lambda^2 + x^2 - 1})\vec{n}]$, where $x = \langle \vec{v}, \vec{n}\rangle$. Notice that if $\lambda < 1$ this implies the restriction $\cos^2\theta_i > 1 - \lambda^2$, so the incident ray can't be far from the normal (otherwise we have 'total reflection'.) Also, the obvious fact that $\vec{w} = \vec{v}$ if $\lambda = 1$ follows from this answer (which is one way to choose the sign of the square root.)