

Answers and comments- Problems from Chapter 9.

9.5 This is a radial Cauchy problem in \mathbb{R}^3 , so $v(r, t) = ru(r, t)$ solves the 1-dimensional wave equation on the half-line $r > 0$, with Cauchy data: $v(r, 0) = 2r$, $v_t(r, 0) = r + r^3$. These are odd functions of r (luckily), so to get $v(r, t)$ to vanish at $r = 0$, we blindly apply d'Alembert's formula as if the data were already defined on all of \mathbb{R} :

$$v(r, t) = \frac{1}{2}[2(r+t) + 2(r-t)] + \frac{1}{2} \int_{r-t}^{r+t} (s + s^3) ds.$$

This leads quickly to the answer:

$$u(r, t) = 2 + t(1 + r^2) + t^3.$$

9.6 Here again $u(r, 0)$ and $u_t(r, 0)$ are even functions of r , so we can assume they are defined on all of \mathbb{R} and apply d'Alembert's formula directly. The answer is:

$$u(r, t) = e^{-(r^2+t^2)} \left[a \cosh(2rt) + \left(t + \frac{b}{2} \right) \frac{\sinh(2rt)}{r} \right].$$

9.8

$$u_{mn}(x, y) = \cos(mx/a) \cos(ny/b), \quad \lambda_{mn} = \frac{m^2}{a^2} + \frac{n^2}{b^2}, \quad m, n = 0, 1, 2, \dots$$

$$f(x, y) \sim A_0 + \sum_{m=1}^{\infty} A_m \cos(mx/a) + \sum_{n=1}^{\infty} \alpha_n \cos(ny/b) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \cos(mx/a) \cos(ny/b).$$

9.10 (a) Fix an integer $q > 2$, and let $n = q + 1$. Obviously n and q have different parity, so $n^2 - q^2$ is odd, and setting $m = (n^2 - q^2 - 1)/2 \geq 1$, we have: $(m+1)^2 + q^2 = m^2 + n^2$. If $m = n$, it follows that $n^2 - 2n = q^2 + 1$, or $(n-1)^2 = q^2 + 2$, impossible since $n-1 = q$. Hence $m \neq n$. Also, since $m+1 = q^2 - q + 1$, $m+1 = n = q+1$ would imply $m = q$, or $q^2 = 2q$, impossible since $q > 2$. Hence $m+1 \neq n$. Thus the linearly independent functions:

$$\sin(m+1)x \sin qy, \quad \sin mx \sin ny, \quad \sin nx \sin my$$

(we take the square $[0, \pi]^2$) all have the same eigenvalue. Since q can be taken arbitrarily large, this happens infinitely many times. (Of course, this argument is just a minor refinement of the remarks on p.247).

(b) If $\frac{m^2}{a^2} + \frac{n^2}{b^2} = \frac{p^2}{a^2} + \frac{q^2}{b^2}$ with $q \neq n$, it follows that $a^2/b^2 = (m^2 - p^2)/(q^2 - n^2)$, which is a rational number.

9.11

$$\begin{aligned}
 2\pi[xJ_{n+1}(x) - nJ_n(x) + xJ'_n(x)] &= \int_0^{2\pi} e^{ix \sin \theta} (xe^{-i\theta} - n + ix \sin \theta) e^{-in\theta} d\theta \\
 &= \int_0^{2\pi} e^{ix \sin \theta} (x \cos \theta - n) e^{-in\theta} d\theta \\
 &= -i \int_0^{2\pi} \frac{d}{d\theta} (e^{ix \sin \theta}) e^{-in\theta} d\theta - n \int_0^{2\pi} e^{ix \sin \theta} e^{-in\theta} d\theta \\
 &= i \int_0^{2\pi} e^{ix \sin \theta} (-in) e^{-in\theta} d\theta - n \int_0^{2\pi} e^{ix \sin \theta} e^{-in\theta} d\theta = 0,
 \end{aligned}$$

integrating by parts (there is no boundary term, since the functions are periodic.)