Problems on Variational properties of eigenvalues- answers.

1. (a) standard calculus; (b) using the hint:

$$\int_{-\pi}^{\pi} 2uu' \frac{\cos x}{\sin x} = \int_{-\pi}^{\pi} u^2 \frac{1}{\sin x} dx.$$

The rest follows easily. (c) Note that $dx = (L/2\pi)dy$ and $f'(x) = (2\pi/L)(du/dy)$. Hence:

$$\int_a^b f^2(x)dx = \frac{L}{2\pi}\int_{-\pi}^{\pi} u^2(y)dy \leq \frac{L}{2\pi}\int_{-\pi}^{\pi} (\frac{du}{dy})^2(y)dy = \frac{L}{2\pi}\int_{-\pi}^{\pi} (L/2\pi)^2 (f')^2 (\frac{2\pi}{L}dx),$$

using the result from part (b) for the inequality. This concludes the proof.

2. (a) Note that, for any $f \in C^2(D)$, the function $f - \bar{f}$ satisfies:

$$\nabla(f - \bar{f}) = \nabla f \quad \text{and} \quad \int_D (f - \bar{f})dx = 0.$$

Since λ_2^N is the minimum value of the Rayleigh quotient over all C^2 functions in D with zero integral over D, if follows that:

$$\lambda_2^N \leq \frac{\int_D |\nabla f|^2 dx}{\int_D (f - \bar{f})^2 dx},$$

for any $f \in C^2(D)$, as we wished to show.

(b) Assume $\Delta u + \lambda_D^2 u = 0$ in D, with $\partial u/\partial n = 0$ on ∂D. Multiply by $u - \bar{u}$ and use Green's identity (the boundary term vanishes due to the Neumann BC):

$$\int_D |\nabla u|^2 dx = \lambda_2^N \int_D (u - \bar{u})u dx = \lambda_2^N \int_D (u - \bar{u})^2 dx,$$

since $\int_D (u - \bar{u})\bar{u} dx = 0$.

The arguments for the Dirichlet eigenvalues are similar (and easier). Part (c) is totally standard.