TIME DECAY FOR THE WAVE EQUATION.

Our goal here is to use the representation formulas for solutions of the Cauchy problem for the wave equation in $\mathbb{R}^n$ ($n = 1, 2, 3$) to derive time decay and stability estimates.

We assume throughout the Cauchy data $f(x) = u(x, 0)$ and $g(x) = u_t(x, 0)$ have *compact support*, that is, are zero outside the ball $B_R = \{x \in \mathbb{R}^n; |x| < R\}$ (for some $R > 0$.)

Notation: $||f|| = \max_{x\in B_R} |f(x)|$, $\quad ||f||_{C^1} = \max_{x\in B_R}(|f(x)|+|\nabla f(x)|)$.

Case 1: $\mathbb{R}$**.** From d'Alembert's formula:

$$u(x, t) = \frac{1}{2}(f(x + t) + f(x - t)) + \frac{1}{2}\int_{x-t}^{x+t} g(s)ds,$$

using the fact that the support of g has length at most $2R$:

$$|u(x, t)| \leq ||f|| + R||g||.$$

This estimate is 'best possible' in general; solutions do not decay in time. The corresponding 'stability estimate' (considering two solutions) is:

$$|u_1(x, t) - u_2(x, t)| \leq ||f_1 - f_2|| + R||g_1 - g_2||.$$

Case 2: $\mathbb{R}^3$**.** In terms of integrals over the unit sphere $S \subset \mathbb{R}^3$, the representation formula is:

$$4\pi u(x, t) = \int_S f(x + t\omega)d\omega + t\int_S (\nabla f \cdot \omega)(x + t\omega)d\omega + t\int_S g(x + t\omega)d\omega.$$

Passing to the sphere $S_x(t)$ with center x, radius t ($dA = t^2 d\omega$) we have:

$$4\pi u(x, t) = \frac{1}{t^2}\int_{S_x(t)} f(y)dA + \frac{1}{t}\int_{S_x(t)} (\nabla f \cdot \omega)(y)dA + \frac{1}{t}\int_{S_x(t)} g(y)dA.$$

Now use the fact that the intersection of $S_x(t)$ with the support of the data B_R has area at most $4\pi R^2$ to obtain the estimate:

$$|u(x, t)| \leq \frac{R^2}{t^2}||f|| + \frac{R^2}{t}||\nabla f|| + \frac{R^2}{t}||g||,$$

so (assuming $t \geq 1$) the solution decays like $1/t$:

$$|u(x, t)| \leq \frac{R^2}{t}(||f||_{C^1} + ||g||).$$

The corresponding 'stability estimate' is:

$$|u_1(x,t) - u_2(x,t)| \leq \frac{R^2}{t}(||f_1 - f_2||_{C^1} + ||g_1 - g_2||).$$

Case 3: $\mathbb{R}^2$**.** The representation formula is:

$$u(x,t) = \frac{1}{2\pi}\frac{\partial}{\partial t}(\int_{B_t(x)} \frac{f(y)dA}{\sqrt{t^2 - |x-y|^2}}) + \frac{1}{2\pi}\int_{B_t(x)} \frac{g(y)dA}{\sqrt{t^2 - |x-y|^2}}.$$

To take the t derivative, we change the term involving f to an integral over the unit disk in $\mathbb{R}^2$ (set $y - x = tz$):

$$\frac{\partial}{\partial t}(\int_{|z|<1} \frac{f(x+tz)t^2dz}{t\sqrt{1-|z|^2}}) = \int_{|z|<1} \frac{f(x+tz)dz}{\sqrt{1-|z|^2}} + t\int_{|z|<1} \frac{(\nabla f \cdot z)dz}{\sqrt{1-|z|^2}}.$$

Moving back to the disk $B_t(x)$, we have:

$$u(x,t) = \frac{1}{2\pi t}\int_{B_t(x)} \frac{f(y)dA}{\sqrt{t^2-|x-y|^2}} + \frac{1}{2\pi t}\int_{B_t(x)} \frac{\nabla f \cdot (y-x)dA}{\sqrt{t^2-|x-y|^2}} + \frac{1}{2\pi}\int_{B_t(x)} \frac{g(y)dA}{\sqrt{t^2-|x-y|^2}}.$$

Bearing in mind that the area of the support of f or g is at most πR^2, we see that the first term is bounded above by $R^2||f||/2t^2$, the second by $R^2||\nabla f||/2t$ (note $|y-x| < t$), the third by $R^2||g||/2t$. We conclude (for $t \geq 1$):

$$|u(x,t)| \leq \frac{R^2}{2t}(||f||_{C^1} + ||g||),$$

which means we again have $1/t$ decay of the amplitude. The corresponding stability estimate for two solutions u_1, u_2 is:

$$|u_1(x,t) - u_2(x,t)| \leq \frac{R^2}{2t}(||f_1 - f_2||_{C^1} + ||g_1 - g_2||).$$