The basic eigenfunctions, Green's functions, Poisson kernels.

Eigenfunctions of the unit disk in $\mathbb{R}^2$:

$$u_{nj}(r,\theta) = J_n(z_{nj}r)\cos n\theta, \quad v_{nj}(r,\theta) = J_n(z_{nj}r)\sin n\theta, \quad n \geq 0, j \geq 1.$$

Here z_{nj} is the jth. positive zero of the Bessel function $J_n(x)$ (for Dirichlet BC) or of its derivative $J_n'(x)$ (for Neumann BC; in this case, the constants are also eigenfunctions). The eigenvalues are z_{nj}^2. For *rotationally symmetric* data, only J_0 occurs; $J_0(z_{0j}r)$, $j \geq 1$, is a complete set of eigenfunctions in $[0,1]$. *Asymptotics:*

$$J_n(x) \sim (const)x^n \text{ at } x \sim 0_+; \quad J_n(x) \sim (const.)\frac{1}{\sqrt{x}}\cos(x-\theta_n) \text{ for } x >> 1.$$

Eigenfunctions of the unit ball in $\mathbb{R}^3$:

$$u_{njm}(r,\omega) = \frac{1}{\sqrt{r}}J_{n+\frac{1}{2}}(z_{nj}r)Y_{nm}(\omega), \quad n \geq 0, j \geq 1, m = 1, \ldots, 2n+1.$$

Here the $Y_{nm}(\omega)$ are *spherical harmonics of degree* n (restrictions to the unit sphere of homogeneous harmonic polynomials of degree n in $\mathbb{R}^3$, a space of dimension $2n+1$). The eigenvalue is z_{nj}^2, the $j^{th.}$ positive zero of the Bessel function $J_{n+\frac{1}{2}}(x)$ (or of its derivative, for Neumann BC); it is $2n+1$-fold degenerate.

Note that the $Y_{nm}(\omega)$ are eigenfunctions for the 'spherical Laplacian':

$$\Delta_S Y_{nm} + n(n+1)Y_{nm} = 0, \quad n \geq 0, m = 1, \ldots 2n+1.$$

For rotationally symmetric data, only $n = 0$ occurs, and this simplifies:

$$u_j(r) = \frac{1}{\sqrt{r}}J_{\frac{1}{2}}(z_{0j}r),$$

with eigenvalue z_{0j}^2 (non-degenerate). The Bessel functions of half-integral order are in fact 'elementary' functions, for example:

$$J_{\frac{1}{2}}(x) = (const.)\frac{\sin x}{\sqrt{x}}, \quad J_{\frac{3}{2}}(x) = (const.)\frac{1}{\sqrt{x}}(\frac{\sin x}{x} - \cos x).$$

Homogeneous harmonic functions in $\mathbb{R}^2$ *and* $\mathbb{R}^3$*:*

$$1, \quad r^{\pm n}\cos n\theta, \quad r^{\pm n}\sin n\theta, \quad \ln r$$

(in the plane); for annuli all occur in the expansion, while for disks and exterior problems some will be missing.

$$r^n Y_{nm}(\omega), \quad r^{-n-1}Y_{nm}(\omega), \quad r^{-1}$$

(in $\mathbb{R}^3$); $n \geq 0$, $m = 1, \ldots 2n+1$.

Green's functions and Poisson kernels, upper half-plane/half-space:

$$G_U(x;y) = \frac{1}{2\pi}(\ln|x-y| - \ln|x-\bar{y}|), \quad x_2 > 0, y_2 > 0.$$

$$P_U(x;y) = \frac{x_2}{\pi}\frac{1}{|x-y|^2}, \quad x_2 > 0, y_2 = 0.$$

$$G_U(x;y) = -\frac{1}{4\pi}(\frac{1}{|x-y|} - \frac{1}{|x-\bar{y}|}), \quad x_3 > 0, y_3 > 0.$$

$$P_U(x;y) = \frac{x_3}{2\pi}\frac{1}{|x-y|^3}, \quad x_3 > 0, y_3 = 0.$$

(Here $y \mapsto \bar{y}$ is the reflection on the boundary.)

Green's functions and Poisson kernels, unit disk/unit ball:

Here $x \mapsto x^* = \frac{x}{|x|^2}$ denotes the reflection ('inversion') on the unit circle (resp. unit sphere), defined for $x \neq 0$. It is an involution that exchanges the interior (minus the origin) and the exterior of the unit ball/disk, and has the property: $|x-y| = |y||x-y^*|$ if $|x| = 1$.

$$G_D(x;y) = \frac{1}{2\pi}(\ln|x-y| - \ln(|y||x-y^*|)), \quad 0 < |y| < 1, |x| < 1;$$

$$G_D(x;0) = \frac{1}{2\pi}\ln|x|, \quad x \neq 0.$$

$$P_D(x;y) = \frac{1-|x|^2}{2\pi}\frac{1}{|x-y|^2}, \quad |x| < 1, |y| = 1.$$

$$G_B(x;y) = -\frac{1}{4\pi}(\frac{1}{|x-y|} - \frac{1}{|y||x-y^*|}), 0 < |y| < 1, |x| < 1;$$

$$G_B(x;0) = -\frac{1}{4\pi|x|} + \frac{1}{4\pi}, \quad x \neq 0.$$

$$P_B(x;y) = \frac{1-|x|^2}{4\pi}\frac{1}{|x-y|^3}, \quad |x| < 1, |y| = 1.$$