SOME SIMPLE EXAMPLES IN POPULATION BIOLOGY

Ex. 1 Competing species systems

$$\begin{cases} x' &= x(1-x-ay) \\ y' &= 2y(1-y-bx) \end{cases}$$

$x(t), y(t)$ are two populations, each with logistic growth in the absence of the other. Presence of both in the environment decreases the growth rate of each by an amount proportional to the other population (the 'interaction strengths' a, b are positive.) There are three cases:

case i: the nullclines for first and second component do not intersect in the first quadrant. This happens when $a > 1, b < 1$. Three equilibria: the origin (a source), (0,1) (a sink) and (1,0) (a saddle). Most solutions have $(0,1)$ as their final (asymptotic) state; this means only the species y survives (if both start out positive). The reason is that the interactions have a weaker effect in the growth rate of y than of x. EXAMPLE: $a = 3, b = 0.5$, window $[0,3] \times [0,2]$.

case ii: $a < 1, b < 1$ (say, $a = 0.3, b = 0.5$, window $[0,3] \times [0,5]$. The two nullclines intersect in the first quadrant, and the intersection is a sink, which is the asymptotic state of most solutions. In addition, there are saddles at $(0,1)$ and $(1,0)$ (and a source at the origin). Both species survive, and their populations converge to a stable value (the interaction strengths are weak, relative to the intrinsic growth rates.)

case iii: $a > 1, b > 1$. (Say, $a = 3, b = 2$ in $[0,2] \times [0,2]$.) Now there are sinks at $(1,0)$ and $(0,1)$, plus a saddle point in the interior. For almost every initial condition, only one of the species survives (you'd have to be impossibly lucky to start at one of the stable separatrices of the saddle. In addition, if you start close to the separatrix it is hard to predict *which* species will survive. One more individual or one less at time zero, and the future fate of the whole population is determined! This is a simple example of *sensitive dependence on initial conditions.*

Ex.2 Cooperative species systems.

$$\begin{cases} x' &= rx(1-x+ay) \\ y' &= sy(1-y+bx) \end{cases}$$

This time the interactions between x and y help the growth of each population (which is still limited in the absence of the other species.) With all parameters assumed positive, there are two cases:

case (i): $\frac{1}{a} < b$. The nullclines don't meet in the first quadrant. There is a source at the origin, and saddles at $(1, 0)$ and $(0, 1)$, with their stable separatrices along the axes. All solutions starting with positive populations of both species experience runaway growth (and 'blow up in finite time'). Example: $a = 2, b = 2, r = 1, s = 2$. Window: $[0, 3] \times [0, 3]$.

case (ii: $\frac{1}{a} > b$. (Example: $a = 0.5, b = 1$ in $[0, 7] \times [0, 7]$.) In addition to the equilibria in case (i), now the nullclines meet in the interior of the first quadrant, and the corresponding equilibrium is a *sink*: we have asymptotic convergence to a stable population for both species (coexistence). Interpretation: since $ab < 1$, at least one of a or b is smaller than 1; in that species, the self-limitation on growth is stronger than the effect of cooperation, so the population remains bounded; this keeps also the other population in check.

Ex.3 Predator-prey with limited growth.

$$\begin{cases} x' &= x(1 - x - ay) \\ y' &= y(1 - y + bx) \end{cases}$$

All parameters are taken positive; not this model assumes that also the predator species has *positive* intrinsic growth rate. We look at four examples.

i) $a = 0.5, b = 2, r = s = 1$ in $[0, 2] \times [0, 3]$. When $1/a > 1$, the nullclines intersect in the open first quadrant (at a sink). The other two equilibria are the origin (a source) and saddles at $(1, 0)$ and $(0, 1)$. The sink is the asymptotic state of a generic initial population (coexistence).

ii) $a = 2, b = 3$ in $[0, 2] \times [0, 3]$. When $1/a < 1$, the nullclines to not intersect in the first quadrant. There is a sink at $(0, 1)$ and a saddle at $(1, 0)$ (with stable separatrices along the x-axis. For any positive starting populations, eventually only the predator survives (not realistic, but results from the positive intrinsic growth rate for the predator species, which presumably is an optional herbivore).

iii) $b = 2, a = 1$: there is a sink in the open first quadrant and a saddle point at $(1, 0)$; we have coexistence at a stable value of the populations.

iv) $b = 0.1, a = 1$. Now there are no equilibria in the open first quadrant, and a sink at $(1, 0)$. Only the prey species survives. When $1/b > 1$, the effect of the interactions on the prey population is week; the predator species is not very good at its hunting job, and eventually dies out.