HISTORICAL INTRODUCTION

Already in the early development of the Differential and Integral Calculus, many applications led to a class of integrals with simple integrands, for which all efforts to represent the integral in closed form through elementary transcendentals proved futile. Among the questions leading to such integrals was the problem of computing the length of an arbitrary elliptical arc.

If a and b are the large and small half-axes of the ellipse, so that its equation referred to the principal axes as coordinate axes is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

in order to find the length of an arbitrary elliptical arc it is enough to consider the lengths of arcs beginning at an endpoint of the small axis and not greater than an elliptical quadrant, that is, the integrals:

$$s = \int_0^x \sqrt{dx^2 + dy^2} = \int_0^x \sqrt{\frac{a^2 - \frac{a^2-b^2}{a^2}x^2}{a^2 - x^2}}dx = a\int_0^{\frac{x}{a}} \sqrt{\frac{1-k^2t^2}{1-t^2}}dt.$$

Here we have set:

$$k^2 = \frac{a^2 - b^2}{a^2}.$$

Clearly $0 < k^2 < 1$.

If we had started from the well-known parametrization of the ellipse:

$$x = a\sin\varphi, \quad y = b\cos\varphi,$$

we would have found:

$$s = \int_0^{\varphi} \sqrt{dx^2 + dy^2} = \int_0^{\varphi} \sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}d\varphi = a\int_0^{\varphi} \sqrt{1 - k^2\sin^2\varphi}d\varphi, \tag{1}$$

where $k^2 = (a^2 - b^2)/a^2$.

These two expressions for s are only formally different, as the substitution $t = \sin\varphi$ takes one into the other. The last integral in (1) is not representable by elementary transcendentals, and must therefore be a new transcendental. The same holds for the other integrals in the same class, which were given the name *elliptic integrals*, after the simplest known example.

Thus we see that the consideration of these new 'elliptical' transcendentals followed inevitably from the needs of the integral calculus. Had the elementary transcendentals not already been known when the integral calculus was invented, it would also have led inevitably to them, as one would have soon realized that integrals such as, for example:

$$\int \frac{dx}{x}, \quad \int \frac{dx}{1+x^2}, \quad \int \frac{dx}{\sqrt{1-x^2}}$$

cannot be expressed using only algebraic functions, and would then have investigated more closely these integrals and their inverse functions.

With the help of elementary transcendentals, in fact even with the help only of the function $\log x$ (or of the functions $\log x$ and $\arctan x$, if one does not wish to leave the domain of the real numbers), it is well-known one can integrate any rational function. In addition, one masters with them the integrals of all functions where the only irrationalities present are square roots of a polynomial of degree one or two, that is, having the form:

$$R(x, \sqrt{ax^2 + 2bx + c}),$$

where R (here and in the sequel) denotes a rational function of two variables. But when the degree of the polynomial under the radical is greater than two, or when the root is no longer a square root, the integration of such an expression in closed form using only elementary transcendentals is, in general, impossible. We must therefore deal with new transcendentals, essentially different from the elementary ones. In particular, if the integral has the form:

$$\int R(x, \sqrt{a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4})dx \tag{2}$$

(where we allow $a_0 = 0$, provided $a_1 \neq 0$), that is, if the only irrationality present is the square root of a polynomial of degree 3 or 4 with only simple zeroes, or if the integral is reducible to this form, then we call it an elliptic integral. These elliptic integrals lead, as we will see, to elliptic functions, of which the characteristic property is the double periodicity.

Treatments of elliptic differentials, meaning differentials such as those appearing in the integral (2), are already found in *J. Wallis* (1616-1703), and also in the work of *Jakob* (1654-1705) and *Johann Bernouilli* (1667-1748), as well as in the *Produzioni matematiche* of Count *G.C. di Fagnano* (1682-1766), to say nothing of other precursors. But it was first *L.Euler* (1707-1783), under the influence of Fagnano's work, who achieved with the discovery of the fundamental addition theorems for elliptic integrals the beginnings of their theory. It was thus with good reason that another founder of the theory, *C.G.J Jacobi* (1804-1851), named the 23 of December, 1751- the day on which *L.Euler* received from the Berlin Academy the assignment to report on the memoir sent by *Fagnano*- the birthday of the elliptic functions.

Shortly after Euler one has to mention *A.M.Legendre* (1752-1833). His main achievement was to have recognized that the newly introduced investigations involved at their core themes of great importance in analysis. With untiring work, to which he dedicated half his life, he sought to develop them, and to build them into a systematic structure. For this he used the fundamental fact, which he discovered, that the integrals (2), through rational transformations, may be brought into three well-defined canonical types. These three integrals, to which all elliptic integrals can be reduced, and in the second of which we recognize the integral to which we were led in the computation of elliptical arc length, are, in *Legendre*'s notation:

$$\begin{aligned} F(\varphi,k) &= \int_0^{\varphi} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}, \\ E(\varphi,k) &= \int_0^{\varphi} \sqrt{1-k^2\sin^2\varphi}\,d\varphi, \\ \Pi(\varphi,n,k) &= \int_0^{\varphi} \frac{d\varphi}{(1+n\sin^2\varphi)\sqrt{1-k^2\sin^2\varphi}}. \end{aligned} \tag{3}$$

Today they are called Legendre normal elliptic integrals of the first, second and third kinds. With the substitution $x = \sin\varphi$, they take the form:

$$\begin{aligned} &\int_0^{x} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \\ &\int_0^{x} \sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx, \\ &\int_0^{x} \frac{dx}{(1+nx^2)\sqrt{(1-x^2)(1-k^2x^2)}}. \end{aligned} \tag{4}$$

It should be noted that nowadays the 'trigonometric forms' (4) of the three normal elliptic integrals, preferred by *Legendre*, are considered unnatural from the standpoint of the general theory, and the form (3) is strongly preferred. All the same, knowledge of the trigonometric forms of the integrals and their properties remains essential, especially for practical computations, since the tables of integrals of first type $F(\varphi, k)$ and second type $E(\varphi, k)$, computed by *Legendre*, refer to them.

Legendre collected his investigations in his *Traité des fonctions elliptiques et des intégrales eulériennes* (2 vols., Paris 1825-1826 and Supplement 1828), where he brought them into organic connection with the work of his predecessors. He seems to have held the opinion that through his work the theory of elliptic integrals (which *Legendre* called *fonctions elliptiques*, contrasting with modern usage) had reached its final form. In fact, the decisive and most fruitful step remained to be taken.

This was done shortly after the publication of *Legendre*'s Traité independently by two brilliant young researchers, the already introduced *Jacobi* and *N.H.Abel* (1802-1829). Both had the idea of considering instead of the elliptic integral of first kind its inverse function, at the same time removing the limitation of considering only real variables, and passing to the complex domain. This inverse function is the simplest 'elliptic function'. Both men discovered more or less simultaneously the main properties of elliptic functions, foremost among them their double periodicity (*Abel* 1827). Even though *Abel* and *Jacobi* hold the priority of publication for these discoveries, the priority for the discoveries themselves belongs to *C.F.Gauβ* (1777-1855). Already in the last years of the 18th. century, *Gauβ* had not only arrived at the results of *Abel* and *Jacobi*, but ventured much farther into new territory. However, of his discoveries Gauβ published no more than brief announcements. Although these announcements gave *Abel* and *Jacobi* the impetus for their investigations, this restraint on the part of *Gauβ* had the consequence that the living development of the theory of elliptic functions was connected to them, and not to him. This explains much of the naming of results in current usage.

To explain the importance of the decisive step taken by *Abel* and *Jacobi*, of passing to the inverse function of an elliptic integral of the first kind, one usually recalls the example of the function $\sin x$, which has much simpler properties than the integral:

$$\text{arc sin } x = \int_0^x \frac{dx}{\sqrt{1-x^2}},$$

of which it is the inverse function. Above all, the sine is single-valued, while the arc sine is infinitely multiple-valued. Returning to the thoughts mentioned previously, had the elementary transcendentals been originally introduced in a manner analogous to the elliptic transcendentals, it would have no doubt seemed useful to investigate, as *Gauβ* and *Abel* did, the inverse functions of these integrals, instead of the integrals themselves, as did *Legendre*.

Thus *Jacobi* bases the theory of elliptic functions not on *Legendre*'s integrals, but rather on the inverse function:

$$\varphi = \text{ am } u$$

of the integral of first type:

$$u = \int_0^{\varphi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}$$

and the three single-valued functions built from this inverse function (am u itself is multivalued):

$$\sin \varphi, \quad \cos \varphi, \quad \Delta\varphi = \sqrt{1 - k^2 \sin^2 \varphi},$$

that is, the functions:

$$\sin \text{ am } u, \quad \cos \text{ am } u, \quad \Delta \text{ am } u.$$

It is simpler to obtain the first function as the inverse function of:

$$u = \int_0^{x} \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}},$$

while the other two are none other than $\sqrt{1 - x^2}$ and $\sqrt{1 - k^2 x^2}$ expressed through u. In effect $\varphi =$ am u never appears by itself in the theory, so the path passing from the trigonometric form of the integral of first kind to its inverse function am u proved to be a detour. Nowadays the simpler notation for the three Jacobi functions sin am u, cos am u, Δ am u:

$$\text{sn } u, \quad \text{cn } u, \quad \text{dn } u,$$

which we'll also use, has become standard.

Of these functions *Jacobi* shows that, among many other properties, also the fundamental one of double periodicity holds. This is the content of the formulas:

$$\begin{aligned} &\text{sn } (u + 4m\mathbf{K} + 2ni\mathbf{K}') = \text{ sn } u, \\ &\text{cn } [u + 4m\mathbf{K} + 2ni(\mathbf{K} + i\mathbf{K}')] = \text{ cn } u \\ &\text{dn } (u + 2m\mathbf{K} + 4ni\mathbf{K}') = \text{ dn } u, \end{aligned} \tag{5}$$

where m and n denote arbitrary integers and $\mathbf{K}$ and $\mathbf{K}'$ are given by:

$$\mathbf{K} = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}, \quad \mathbf{K}' = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - (1 - k^2) \sin^2 \varphi}}.$$

Jacobi then shows how, with the help of these three single-valued meromorphic functions- or, more precisely, with the help of certain entire transcendental functions, the so-called theta functions, from whose ratios sn u, cn u and dn u can be represented- the normal integrals of *Legendre* can easily be computed, thereby solving all problems whose solutions lead to them.

The most important contributions to the theory of elliptic functions after *Abel* and *Jacobi* were doubtlessly made by *K. Weierstraβ* (1815-1897). He replaced the functions sn u, cn u and *dn* u by the single function $\mathcal{P}\ u$, which in many respects behaves in a simpler way than those. Likewise, he replaced the Jacobi theta functions by other functions, the sigma functions, which exhibit analogous advantages.

The simplicity, clarity and completeness of the whole structure of the theory thus obtained offers such substantial advantages, that also we, following the example of many authors, will start from the Weierstraβ $\mathcal{P}$-functions, even though this means traversing the route of the historical development backwards. Only later do we pass to Jacobi's functions and the theta functions. While the behavior of these functions is less simple than the behavior of the Weierstraβ functions, they are still to be preferred in numerous computations.

In conclusion we remark that integrals involving an irrationality of higher order than we have allowed, for example the square root of a polynomial of degree 5 or higher ('hyperelliptic', and even more generally 'Abelian' integrals), do not lead to a theory corresponding entirely to the theory of elliptic functions. The inverse functions must then have more than two periods, and therefore, as we shall see, cannot be single-valued. Nevertheless the theory was successfully extended to these integrals- again by *Abel* and *Jacobi*- but the functions one obtains (hyperelliptic and Abelian functions) are functions of several [complex] variables. They do exhibit an extremely important property (Abel's theorem) which is a generalization of the addition theorem for elliptic integrals; however, they have so far remained of lesser importance for applications than the elliptic functions.

(Francesco Tricomi, Elliptische Funktionen, Akademische Verlagsgesellschaft, Leipzig 1948-p.1-7, translated from the German by A.F.)