MATH 251- EXAM 3- November 21, 2005

1. Let R be the rotation in $\mathbb{R}^3$ with axis direction $(1,3,1)$, angle $\pi/4$ (counterclockwise when looking down the axis.)

(a) Find an orthonormal basis $\mathcal{B} = \{u, v_1, v_2\}$ of $\mathbb{R}^3$, where u is on the axis and $\{v_1, v_2\}$ is an orthonormal basis for the plane orthogonal to the axis;

Pick two vectors in the plane $x_1+3x_2+x_3 = 0$ perpendicular to the axis; say, $w_1 = (3,-1,0)$ and $w_2 = (1,0,-1)$. Passing to an orthogonal basis, we set $\hat{v_1} = w_1$ and replace w_2 by:

$$\hat{v_2} = w_2 - \frac{\langle w_1, w_2\rangle w_1}{||w_1||^2} = (1,0,-1) - \frac{3}{10}(3,-1,0) = \frac{1}{10}(1,3,-10).$$

Normalizing these vectors we obtain the orthonormal basis:

$$u = \frac{1}{\sqrt{11}}(1,3,1), \quad v_1 = \frac{1}{\sqrt{10}}(3,-1,0) \quad v_2 = \frac{1}{\sqrt{110}}(1,3,-10).$$

(It is easy to check that, in this order, this basis is positive.)

(b) Write down the matrix $[R]_{\mathcal{B}}$ of R in the basis $\mathcal{B}$.

$$[R]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}.$$

(c) Find an orthogonal matrix P so that the matrix of R in the standard basis of $\mathbb{R}^3$ is $P[R]_{\mathcal{B}}P^T$.

$$P = [u|v_1|v_2] \quad \text{(by columns)}$$

2. For the matrix A given below, find the eigenspace for the eigenvalue 2, and explain why A is not diagonalizable.

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 2 & 0 \\ -2 & 5 & 2 \end{bmatrix}.$$

$$(A-2I)v = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 0 & 0 \\ -2 & 5 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3x \\ x \\ -2x+5y \end{bmatrix} = 0.$$

This gives immediately: $x = y = 0$, z arbitrary, so:

$$E(2) = \{t(0,0,1); t \in \mathbb{R}\}.$$

A is triangular, so 5 and 2 are the only eigenvalues, the latter with algebraic multiplicty 2, but only a one-dimensional eigenspace. Since the dimension of $E(5)$ is also 1, $\mathbb{R}^3$ does not admit a basis consisting of eigenvectors of A.

3. The line $y = ax+b$ is the least-squares fit to the points $(-1,1), (0,2), (1,4), (2,4)$. (a) Write down the 'normal system' for the problem (a 2×2 system for the vector (b, a)).

$$A^T A \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix} = A^T \begin{bmatrix} 1 \\ 2 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 11 \\ 11 \end{bmatrix}.$$

(A is the 'design matrix' given in part (b))

(b) The 'design matrix' for this problem is: $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$. Find the orthogonal projection of $(1,2,4,4)$ on the column space of A (*Hint:* solve the normal system and use the equation of the line.)

Solving the normal system by inverting $A^T A$, we find $b = 2.2, a = 1.1$. Substituting $x = -1, 0, 1, 2$ in the equation $y = 1.1x + 2.2$, we find $y = 1.1, 2.2, 3.3, 4.4$, so the projection is $(1.1, 2.2, 3.3, 4.4)$.

4. The matrix A given below has eigenvalues $2 \pm i$. Find the standard form $\Lambda = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ of A, and a matrix V so that $V^{-1}AV = \Lambda$.

In the usual way, we find the eigenvector $(1, 2+i)$ for the eigenvalue $2+i$. Its real and imaginary parts are the column vectors of V:

$$V = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \quad \Lambda = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}.$$

5. (a) Find a 2×2 matrix P and a diagonal matrix Λ so that $P^{-1}BP = \Lambda$. (B is given below.)

The characteristic polynomial of A is $\lambda^2 - 3\lambda + 2$, with roots 1 and 2, the eigenvalues of B. Proceeding as usual, we find $(1,1)$ (resp. $(1,2)$) as an *eigenvector for eigenvalue 1 (resp. eigenvalue 2). Thus Λ and P are:*

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

(b) Find the vector $B^n \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ explicitly (as a function of n). (Note that $(1, 2)$ is an eigenvector of B.)

Since $(1, 2)$ is an eigenvector of B with eigenvalue 2:

$$B^n \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2^{n-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2^{n-1} \\ 2^n \end{bmatrix}.$$

$$A = \begin{bmatrix} 0 & 1 \\ -5 & 4 \end{bmatrix} \text{ (Problem 4) } B = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \text{ (Problem 5).}$$