

Math 241, Spring 2006, NAME:

List 8: Divergence and curl of vector fields, theorems of Stokes and Gauss, physics applications- DUE FRIDAY, 4/28

1. Transform the flux of the curl of the given vector field on the given surface to a line integral using Stokes' theorem, then compute the integral.

$$(i)\mathbf{F}(x, y, z) = (y, z, x);$$

S : $z \geq 0$ portion of the paraboloid $z = 1 - x^2 - y^2$, oriented by the upward normal.

$$(ii)\mathbf{F}(x, y, z) = (xz, -y, x^2y),$$

S : tetrahedron bounded by the coordinate planes and the plane $3x + y + 3z = 6$, except for the face on the xz plane, oriented by the outward normal.

2. Use Stokes' theorem to show the line integrals have the values given (specify how C must be oriented.)

(i) $\int_C (y + z)dx + (z + x)dy + (x + y)dz = 0$, where C is the intersection of the cylinder $x^2 + y^2 = 2y$ and the plane $y = z$.

(ii) $\int_C (y - z)dx + (z - x)dy + (x - y)dz = 2\pi a(a + b)$, where C is the intersection of the cylinder $x^2 + y^2 = a^2$ and the plane $x/a + z/b = 1$, where $a > 0, b > 0$.

3. Let $\mathbf{F} = (-y/(x^2 + y^2), x/(x^2 + y^2), z)$, defined on the torus S obtained by rotating the circle $(x - 2)^2 + z^2 = 1$ (in the xz plane) about the z axis. Show that the curl of \mathbf{F} is zero on S , but the line integral of \mathbf{F} along the circle $x^2 + y^2 = 9, z = 0$ is not zero. Why doesn't this contradict Stokes' theorem?

4. For each \mathbf{F} given below, can there be a vector field \mathbf{G} so that $\mathbf{F} = \text{curl } \mathbf{G}$?

$$(i)\mathbf{F} = (xy^2, yz^2, zx^2); \quad (ii)\mathbf{F} = (yz, xyz, xy).$$

5. Prove the following vector identities (for arbitrary functions u , vector fields \mathbf{F}) [choose two and prove them]:

$$\text{div}(u\mathbf{F}) = u(\text{div}\mathbf{F}) + \nabla u \cdot \mathbf{F}$$

$$\text{curl}(u\mathbf{F}) = u(\text{curl}\mathbf{F}) + \nabla u \times \mathbf{F}$$

$$\text{div}(\mathbf{F} \times \mathbf{G}) = (\text{curl}\mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\text{curl}\mathbf{G})$$

$$\text{curl}(\text{curl } \mathbf{F}) = \nabla(\text{div}\mathbf{F}) - \Delta\mathbf{F}$$

6. Use the divergence theorem to compute the outward flux of \mathbf{F} across the boundary of the given region D .

$$(i)\mathbf{F} = (rx, ry, rz), \quad D : 1 \leq r^2 \leq 4.$$

$$(ii) \mathbf{F} = (y, xy, -z), \quad D : x^2 + y^2 \leq 4, 0 \leq z \leq x^2 + y^2.$$

7. Use the divergence theorem to show that the volume of the region D bounded by a closed surface S in \mathbb{R}^3 equals one-third of the flux of $\mathbf{F}=(x,y,z)$ across S .

8.(i) Use the divergence theorem to evaluate

$$\int \int_S (2x + y^2 + 3z) dA,$$

where S is the sphere $x^2 + y^2 + z^2 = 4$.

(ii) Use the divergence theorem to evaluate the flux of the vector field given below on the upper unit hemisphere (with upward normal orientation), by transferring it to an integral over the unit disk in the plane $z = 0$.

$$\mathbf{F} = (x^2 + yz^5, x^3 + y^2 + \sin z, x^2 + y^2).$$

(note that here the divergence of F is not zero, but maybe its integral over an appropriate region is.)