

Math 241, Spring 2006, NAME:

List 8: Divergence and curl of vector fields, theorems of Stokes and Gauss, physics applications

1. Transform the flux of the curl of the given vector field on the given surface to a line integral using Stokes' theorem, then compute the integral.

$$(i)\mathbf{F}(x, y, z) = (y^2, xy, xz)$$

S : upper unit hemisphere, oriented by the upward normal.

$$(ii)\mathbf{F}(x, y, z) = (y, z, x);$$

S : $z \geq 0$ portion of the paraboloid $z = 1 - x^2 - y^2$, oriented by the upward normal.

$$(iii)\mathbf{F}(x, y, z) = (xz, -y, x^2y),$$

S : tetrahedron bounded by the coordinate planes and the plane $3x + y + 3z = 6$, except for the face on the xz plane, oriented by the outward normal.

2. Use Stokes' theorem to show the line integrals have the values given (specify how C must be oriented.)

(i) $\int_C ydx + zdy + xdz = \pi a^2/3$, where C is the intersection of the sphere $x^2 + y^2 + z^2 = a^2$ and the plane $x + y + z = 0$.

(ii) $\int_C (y + z)dx + (z + x)dy + (x + y)dz = 0$, where C is the intersection of the cylinder $x^2 + y^2 = 2y$ and the plane $y = z$.

(iii) $\int_C (y - z)dx + (z - x)dy + (x - y)dz = 2\pi a(a + b)$, where C is the intersection of the cylinder $x^2 + y^2 = a^2$ and the plane $x/a + z/b = 1$, where $a > 0, b > 0$.

3. Let $\mathbf{F} = (-y/(x^2 + y^2), x/(x^2 + y^2), z)$, defined on the torus S obtained by rotating the circle $(x - 2)^2 + z^2 = 1$ (in the xz plane) about the z axis. Show that the curl of \mathbf{F} is zero on S , but the line integral of \mathbf{F} along the circle $x^2 + y^2 = 9, z = 0$ is not zero. Why doesn't this contradict Stokes' theorem?

4. For each \mathbf{F} given below, can there be a vector field \mathbf{G} so that $\mathbf{F} = \text{curl } \mathbf{G}$?

$$(i)\mathbf{F} = (xy^2, yz^2, zx^2); \quad (ii)\mathbf{F} = (yz, xyz, xy).$$

5. Prove the following vector identities (for arbitrary functions u , vector fields \mathbf{F}):

$$\text{div}(u\mathbf{F}) = u(\text{div}\mathbf{F}) + \nabla u \cdot \mathbf{F}$$

$$\text{curl}(u\mathbf{F}) = u(\text{curl}\mathbf{F}) + \nabla u \times \mathbf{F}$$

$$\text{div}(\mathbf{F} \times \mathbf{G}) = (\text{curl}\mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\text{curl}\mathbf{G})$$

$$\text{curl}(\text{curl } \mathbf{F}) = \nabla(\text{div}\mathbf{F}) - \Delta\mathbf{F}$$

6. Use the divergence theorem to compute the outward flux of \mathbf{F} across the boundary of the given region D .

$$(i) \mathbf{F} = (x^2, -2xy, 3xz), \quad D : x^2 + y^2 + z^2 \leq 4 \text{ first octant}$$

$$(ii) \mathbf{F} = (rx, ry, rz), \quad D : 1 \leq r^2 \leq 4.$$

$$(iii) \mathbf{F} = (y, xy, -z), \quad D : x^2 + y^2 \leq 4, 0 \leq z \leq x^2 + y^2.$$

7. Use the divergence theorem to show that the volume of the region D bounded by a closed surface S in \mathbb{R}^3 equals one-third of the flux of $\mathbf{F}=(x,y,z)$ across S .

8.(i) Use the divergence theorem to evaluate

$$\int \int_S (2x + 2y + z^2) dA,$$

where S is the sphere $x^2 + y^2 + z^2 = 1$.

(ii) Use the divergence theorem to evaluate the flux of the vector field given below on the upper unit hemisphere (with upward normal orientation), by transferring it to an integral over the unit disk in the plane $z = 0$.

$$\mathbf{F} = (yz^2, x^3 + \tan z, x^2 + y^2).$$

9. Conservation of mass. Let \mathbf{v} be the velocity vector field of a fluid with density $\rho(x, y, z, t)$ (t =time) in a region T bounded by a closed surface S . Show that the conservation of mass equation:

$$\frac{d}{dt} \int \int \int_T \rho dV = - \int \int_S \rho \mathbf{v} \cdot \mathbf{N} dA$$

is equivalent to the *continuity equation*:

$$\operatorname{div}(\rho \mathbf{v}) + \frac{\partial \rho}{\partial t} = 0.$$

10. Maxwell's equations and the wave equation. Maxwell's equations for (time-dependent) electric field \mathbf{E} and magnetic field \mathbf{B} in vacuum read:

$$\operatorname{div} \mathbf{E} = 0, \quad \operatorname{div} \mathbf{B} = 0, \quad \operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \operatorname{curl} \mathbf{B} = \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}.$$

Use the equations and the identities in problem **5** to show each component of \mathbf{E} and of \mathbf{B} satisfies the *wave equation* with speed of propagation $c = (\epsilon_0 \mu_0)^{-1/2}$:

$$u_{tt} - c^2 \Delta u = 0.$$

11. Static electric and magnetic fields The equations for electric (resp. magnetic) fields determined by a static charge (resp, steady current) configuration in vacuum can be stated in differential or in integral form. Use a theorem of vector calculus to pass from the differential to the integral form.

(a) $\operatorname{div} \mathbf{E} = \rho/\epsilon_0$, where ρ =charge density; $\operatorname{div} \mathbf{B} = 0$.

The total outward flux of \mathbf{E} across a closed surface equals $(1/\epsilon_0) \times$ the total charge enclosed by the surface (Gauss law). For the magnetic field \mathbf{B} , the total flux across any closed surface is *zero*;

(b) $\operatorname{curl} \mathbf{E} = 0$ and $\operatorname{curl} \mathbf{B} = \mu_0 \mathbf{j}$, where \mathbf{j} =current density.

Electric fields are conservative (zero circulation around any closed loop); let $\mathbf{E} = -\nabla V$ define the electrostatic potential V . Then $-\Delta V = \rho/\epsilon_0$ (Poisson's equation). The circulation of \mathbf{B} around any closed loop equals $\mu_0 \times$ the total current linked by the loop (Ampère's law)

(c) *conservation of charge* $\frac{\partial \rho}{\partial t} = -\operatorname{div} \mathbf{j}$.

Let S be a closed surface; the time derivative of the total charge enclosed by S equals minus the total current through S (i.e., minus the outward flux of \mathbf{j} through S .)

12. Time-dependent electric-magnetic fields

(a) $\operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$.

A time-dependent magnetic field 'induces' an electric field; the circulation of \mathbf{E} around a closed loop is no longer zero, but instead equals minus the time derivative of the magnetic flux through any surface bounded by the loop (Faraday-Henry law.)

(b) $\operatorname{curl} \mathbf{B} = \mu_0 \mathbf{j} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}$.

A time-dependent electric field 'induces' a magnetic field; in this case the circulation of \mathbf{B} around any closed loop equals $\epsilon_0 \mu_0$ times the time derivative of the electric flux through any surface bounded by the loop *plus* μ_0 times the total current through the *same* surface (Ampère's Law with Maxwell's correction.)

13. Potential fluid flow. Let \mathbf{v} be the velocity vector field of a fluid in a bounded, simply-connected region $D \subset \mathbb{R}^2$. (i) If \mathbf{v} is incompressible (zero divergence) and irrotational (zero curl), then $\mathbf{v} = \nabla u$ for a potential function u , which is harmonic ($\Delta u = 0$) in D . (ii) The total flux of \mathbf{v} through the boundary of D is zero. (iii) If the normal component of \mathbf{v} is zero on the whole boundary, then \mathbf{v} must be zero in D ; to show this, use the divergence theorem for the vector field $u\mathbf{v}$, and the identity:

$$\operatorname{div}(u\mathbf{v}) = \|\mathbf{v}\|^2 + u \operatorname{div} \mathbf{v}.$$