Math 241- EXAM 3- April 19, 2006. Closed book, closed notes. No calculators. 50 min test. Show all work.

1. Consider the mapping of $\mathbb{R}^2$ defined by:

$$x(u,v) = 2uv, \quad y(u,v) = u^2 - v^2.$$

(i)[6] (*list 6 no. 9*) For a smooth function $f(x,y)$ and the composition $g(u,v) = f(x(u,v), y(u,v))$, find expressions for the first derivatives g_u, g_v, in terms of f_x, f_y, u and v.

(ii)[6] *(review handout no. 6)*If f satisfies the Laplace equation $f_{xx}+f_{yy} = 0$, show that g satisfies the Laplace equation as well: $g_{uu} + g_{vv} = 0$.

2.*(list 6 no. 1; review handout no. 8)* Consider the linear change of variable: $u = x + y, v = x - y$ and the double integral:

$$I = \int_0^2 \int_x^{4-x} (x+y)e^{(x-y)}\,dydx.$$

(i)[6] Sketch the region of integration in the xy plane (a triangle) and the corresponding region of integration in the uv plane (another triangle.)

(ii)[6] Write the double integral I in uv coordinates, without computing its value. You will need the Jacobian determinant of the inverse mapping: $x = (1/2)(u+v), y = (1/2)(u-v)$.

3. *(list 6 no. 4)*[7] Use integration in spherical coordinates to find the z-coordinate of the centroid of the half-shell:

$$T = \{(x,y,z) | 1 \leq x^2 + y^2 + z^2 \leq R^2, z \geq 0\}.$$

Note: $vol(T) = (2\pi/3)(R^3 - 1), \quad z = \rho\cos\varphi, \quad dV = \rho^2 \sin\varphi d\rho d\varphi d\theta$

4.[6] *(list 7 no. 5)* Find a potential function for the conservative vector field in $\mathbb{R}^3$:

$$\mathbf{F}(x,y,z) = (x+z, -y-z, x-y).$$

5.[6]*(list 7 no. 7; review handout no. 2)* Use Green's theorem to find the area enclosed by the parametrized simple closed curve:

$$\mathbf{r}(t) = (t^2, t^3 - 3t), \quad t \in [-\sqrt{3}, \sqrt{3}].$$

6.[7]*(list 7 no. 9; review handout no. 1)* A fluid in the plane has unit mass density and velocity vector field $\mathbf{v}(x,y) = (y^2, xy)$. The net mass of fluid leaving the region $\{x^2 + y^2 \leq 1\}$ per unit time is the outward *flux* of $\mathbf{v}$ across the boundary of the region. Use the divergence theorem in the plane to compute this flux.