A NOTE ON THE DEFINITION OF AREA OF A CURVED SURFACE

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We regard the length of a curve as the limiting value of the length of an inscribed polygon as the length of the largest individual side tends to zero. For the measurement of area, a direct analogy would be as follows: in the curved surface we inscribe a polyhedron formed of plane triangles, determine the area of the polyhedron, make the inscribed net of triangles finer by letting the length of the longest side tend to zero, and seek to find the limiting value of the area of the polyhedron. *It turns out, however, that such a definition of area would have no meaning, as in general this process does not yield a definite limiting value.* Below we describe precisely a simple example.

This phenomenon may be explained in the following way: a polygon inscribed in a smooth curve always has the property (which follows from the mean-value theorem)that the direction of the individual side of the polygon approaches the direction of the curve as closely as we please, if the subdivision is fine enough. With curved surfaces the situation is quite different. The sides of a polyhedron inscribed in a curved surface may be inclined relative to the tangent plane of the surface *as steeply as we please*, even if the polyhedral faces have arbitrarily small diameters. The area of such a polyhedron, therefore, will in general be a poor approximation to the area of the curved surface.

In the definition of length of a smooth curve, however, instead of using an inscribed polygon we can equally well use a *circumscribed* polygon, that is, a polygon of which every side is tangent to the curve at exactly one point. This definition of length (as the limit of lengths of circumscribed polygons) can easily be extended to surfaces. For a surface defined as a graph $z = f(x, y)$ over a region R in the xy-plane, we subdivide R into small triangles R_i, choose a point $P_i \in R_i$ and construct the circumscribed polyhedron whose i^{th} face is the part of the tangent plane to the surface at $(P_i, f(P_i))$ above the triangle R_i. Now the limit of the areas of these polyhedra as we take progressively finer subdivisions can be shown to exist (this isn't hard), and gives the formula found in all calculus textbooks.

1 Example

Consider a cylindrical surface in xyz-space with the equation $x^2 + y^2 = 1$, lying between the planes $z = 0$ and $z = 1$. (The area of this surface is, of course, 2π.)In it we now inscribe a polyhedral surface, all of whose faces are congruent isosceles triangles, as follows. We first subdivide the circumference of the circle into n equal parts, and on the cylinder we consider m equidistant circles $z = 0, h, 2h, \ldots, (m-1)h$, where $h = 1/m$. We perform the division of each of these circles into n equal parts, in such a way that the points of division of each circle lie above the midpoints of the arcs of the preceding circle. Consider the polyhedron inscribed in the cylinder whose edges consist of the chords of the circles and lines joining neighboring subdivision points of neighboring circles. All the faces are congruent isosceles triangles, and simple geometry shows that the base b and height h are, respectively:

$$b = 2\sin(\frac{\pi}{n}), \quad h = \sqrt{\frac{1}{m^2} + (1 - \cos(\frac{\pi}{n}))^2} = \sqrt{\frac{1}{m^2} + 4\sin^4(\frac{\pi}{n})}.$$

If n and m are sufficiently large this polyhedron will lie as close as we please to the cylindrical surface. If we now keep n fixed, we can choose m so large that each of the triangles is as nearly parallel as we please to the xy-plane, and therefore makes an arbitrarily steep angle with the surface of the cylinder. Since the total number of triangles is $2nm$, the total area of the polyhedron is:

$$F_{n,m} = 2mn\sin(\frac{\pi}{n})\sqrt{\frac{1}{m^2} + 4\sin^4(\frac{\pi}{n})} = 2n\sin(\frac{\pi}{n})\sqrt{1 + 4m^2\sin^4(\frac{\pi}{n})}.$$

The limit of this expression is *not* independent of the way in which m and n tend to infinity. If, for example, we keep n fixed and let $m \to \infty$, the expression increases beyond all bounds. If we make m and n go to infinity together, setting $m = n$, the expression tends to 2π. If we put $m = n^2$, we obtain the limit $2\pi\sqrt{1 + \pi^4/4}$, and so on. In fact, from the above expression we see that $F_{n,m} \geq 2n\sin(\pi/n)$, so that the limit we obtain when $n = m$ (namely, 2π) is the smallest possible, among all possible ways of letting $m, n \to \infty$. There is a name for this: 2π is the 'lower limit' (or 'limit inferior' of the double sequence $F_{n,m}$ as $n, m \to \infty$. In symbols:

$$\lim \inf_{n,m\to\infty} F_{n,m} = 2\pi.$$

So here is a general fact, of which this example is just a particular instance. If we have any arbitrary sequence of polyhedra tending to a given surface, the areas of the polyhedra may fail to tend to the area of the surface. But if for every such sequence of approximating polyhedra we find the "limit inferior" of their areas, we obtain a set of positive real numbers associated with the surface. *The limit inferior of this set of numbers coincides with the area of the surface* (In fact, this can be taken as the definition of area.) This fact is referred to as 'lower semi-continuity of surface area under polyhedral approximation'.

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