Non-homogeneous 2nd order equations-examples.

The general solution always has the form:

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t),$$

where $c_1 y_1 + c_2 y_2$ is the general solution of the homogeneous problem $L[y] = 0$ and $y_p(t)$ is a fixed but arbitrary solution of the non-homogeneous problem ('particular solution'). In the following we just compute $y_p(t)$.

1. $y'' + y' + y = t^2$. Try $y(t) = at^2 + bt + c$. We find:

$$L[y] = at^2 + (b + 2a)t + (b + 2a + c),$$

which gives $a = 1, b = -2, c = 0$.

2. $y'' - 3y' + 2y = (1+t)e^{3t}$. Here $e^3 t$ is not a solution of the homogeneous problem, so we try $y(t) = (a + bt)e^{3t}$ and compute:

$$L[(a + bt)e^{3t}] = [(9a + 6b + 9bt) - 3(3a + b + 3bt) + 2(a + bt)]e^{3t}.$$

Now match coefficients.

3. $y'' - 3y' + 2y = t\sin(2t)$. Here $2i$ is not a root of the characteristic equation $r^2 - 3r + 2 = 0$, so we try $y = (at + b)\sin 2t + (ct + d)\cos 2t$ and find:

$$L[y] = [(-3a-2b-4c+6d)+(-2a+6c)t]\sin(2t)+[(4a-6b-3c-2d)+(-6a-2c)t]\cos(2t).$$

Match coefficients to obtain an easily solvable linear system for a, b, c, d.

4. $y'' + 4y = \sin(2t)$. Now $2i$ *is* a root of the characteristic equation, so we try: $y(t) = te^{2it}$ and find:

$$L[y] = 4ie^{2it}.$$

Dividing by $4i$ and taking the imaginary part:

$$y(t) = \mathbb{I}m[\frac{te^{2it}}{4i}] = -\frac{t}{4}\cos(2t).$$

5. $y'' + 4y = t\sin(2t)$. We know from example 4 that $L[te^{2it}] = 4ie^{2it}$, so we try instead $y(t) = t^2 e^{2it}$ and compute:

$$L[t^2 e^{2it}] = (2 + 8it)e^{2it}.$$

Hence to get $L[y] = te^{2it}$ we take the linear combination:

$$L[\frac{t^2 e^{2it}}{8i} - \frac{1}{4i}(\frac{1}{4i}te^{2it})] = te^{2it},$$

and taking imaginary part we find a solution:

$$y_p(t) = \mathbb{I}m\{[-i\frac{t^2}{8} + \frac{1}{16}t]e^{2it}\} = -\frac{t^2}{8}\cos(2t) + \frac{t}{16}\sin(2t).$$

6. $4y'' + 4y' + 5y = e^t \cos(2t)$. The roots of the characteristic equation are $-1/2 \pm i$. Since $1 + 2i$ is not one of them, there is a solution of the form $\mathbb{R}e[Ae^{(1+2i)t}]$, for some complex number A. We compute:

$$L[e^{(1+2i)t}] = (-3 + 24i)e^{(1+2i)t},$$

so the real part of $(-3 + 24i)^{-1}e^{(1+2i)t}$ is a solution.

7. $4y'' + 4y' + 5y = 5e^{-t/2}\cos t$. Here $\omega = -1/2 + i$ is a root of the characteristic equation, and the 'forcing term' is the real part of $e^{\omega t}$. So we try $y(t) = te^{\omega t}$ and find:

$$L[te^{\omega t}] = (8\omega + 4)e^{\omega t},$$

hence the real part of $5te^{\omega t}/(8\omega + 4)$ will be a solution:

$$y(t) = \mathbb{R}e[-\frac{i}{8}(5te^{\omega t})] = 40te^{-t/2}\sin t.$$

8. $y'' + 4y' + 4y = e^{-3t}$. Here the homogeneous equation has -2 as a double root. Since -3 is not a root, there will be a solution of the form $(const.)e^{-3t}$, and indeed since $L[e^{-3t}] = e^{-3t}$, the constant is 1.

9. $y'' + 4y' + 4y = e^{-2t}$. Now the exponent in the 'forcing term' coincides with the double root of the homogeneous equation, so it would be pointless to try e^{-2t}, te^{-2t} or any linear combination of these two. The next possibility is $t^2 e^{-2t}$. We find:

$$L[t^2 e^{-2t}] = 2e^{-2t},$$

so $y(t) = (1/2)t^2 e^{-2t}$ is a solution.

10. $y'' + 4y' + 4y = te^{-2t}$. Given the previous example, it would be pointless to try a linear combination of the form $(a + bt + ct^2)e^{-2t}$. Next in line is:

$$L[t^3 e^{-2t}] = 6te^{-2t},$$

so $y(t) = (1/6)t^3e^{-2t}$ is a solution.

The conclusion from examples 9 and 10 is that when the characteristic equation has a double root, which also occurs in the 'exponential part' of the forcing term, we have to increase the degree of the 'polynomial part' by *two* when looking for a solution by 'undetermined coefficients'.