EXACT DIFFERENTIAL EQUATIONS.

Consider the first-order equation:

$$M(t, y(t))y' + N(t, y(t)) = 0, \quad y = y(t).$$

Here M and N are arbitrary functions of two variables, with continuous first-order partial derivatives in t and y.

A function $\varphi(t, y)$ of two variables is a *conserved quantity* for the differential equation if, for any solution $y(t)$, we have: $\varphi(t, y(t)) = const$. Equivalently,

$$\frac{d}{dt}\varphi(t, y(t)) \equiv 0,$$

for any solution $y(t)$.

When this happens, $\varphi(t, y) = C$ is the 'general solution in implicit form' of the DE.

Ex. 1 $2ty^3 + 3t^2y^2y' = 0$, $\varphi(t, y) = t^2y^3$.

Ex. 2 $\cos y - (t \sin y - y^2)y' = 0$, $\varphi(t, y) = 3t \cos y + y^3$.

A first-order differential equation is *exact* if it has a conserved quantity. For example, separable equations are always exact, since by definition they are of the form:

$$M(y)y' + N(t) = 0,$$

and then if $A(y)$, $B(t)$ are antiderivatives of M and N (resp.), this is the same as:

$$(A(y) + B(t))' = 0,$$

so $\varphi(t, y) = A(y) + B(t)$ is a conserved quantity.

How do we recognize whether a differential equation is exact, and how do we find a conserved quantity?

We need the concept of 'partial derivatives' of a function of two variables $f(x, y)$, and the properties:

(i) $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$, if the second-order partial derivatives are continuous;

(ii) *(chain rule)* If $f(x, y)$ has continuous first-order partial derivatives and $g(x)$ is a differentiable function of one variable, define the new function of one variable: $h(x) = f(x, g(x))$. Then h is differentiable and:

$$h'(x) = \frac{\partial f}{\partial x}(x, g(x)) + g'(x)\frac{\partial f}{\partial y}(x, g(x)).$$

The following theorem answers the question above.

Theorem. *Let $M(t,y)$ and $N(t,y)$ have continuous first-order partial derivatives in a rectangle $R = \{(t,y); a < t < b, c < y < d\}$. The differential equation:*

$$M(t,y)y' + N(t,y) = 0$$

admits a conserved quantity $\varphi(t,y)$ in R if, and only if:

$$\frac{\partial M}{\partial t} = \frac{\partial N}{\partial y} \text{ in } R.$$

The fact that the condition is necessary follows from (i) above. To illustrate that it is also sufficient, we show how to find a conserved quantity via 'partial integration'.

Ex.3 $3y + e^t + (3t + \cos y)y' = 0.$

Ex.4 $4t^3e^{t+y} + t^4e^{t+y} + 2t + (t^4e^{t+y} + 2y)y' = 0.$

Remark. The theorem shows that the differential equations for which a 'general solution' can be found, even in implicit form, are rare in the universe of all possible differential equations, since randomly chosen functions $M(t,y)$ and $N(t,y)$ will almost never satisfy the condition in the theorem.

HOMEWORK PROBLEMS

(A)Verify that the following DEs satisfy the criterion for exactness, and find a conserved quantity:

1. $2t\sin y + y^3e^t + (t^2\cos y + 3y^2e^t)y' = 0$

2. $\frac{y^2}{2} - 2ye^t + (yt - 2e^t)y' = 0$

3. $(3e^{3t}y - 2t) + e^{3t}y' = 0$

(B) Solve the following initial-value problems. Whenever possible, find an explicit form $y(t)$ for the solution, including its domain of definition (an *interval* containing t_0.)

4. $2ty^3 + 3t^2y^2y' = 0, \quad y(1) = 1.$

5. $3t^2 + 4ty + (2y + 2t^2)y' = 0, \quad y(0) = 1$

6. $3ty + y^2 + (t^2 + ty)y' = 0, \quad y(2) = 1.$