Math 231.04, Problem Set 8 Solutions (Partial)

Due Wednesday, March 17, 2010

From Text Fundamentals of Differential Equations, by Nagle, Saff, and Snider

Section 4.4, # 11, 13, 15, 17
Section 4.5, # 17, 19, 25

Additional Problems:

1.) a.) Suppose $A, \omega$, and $\phi$ are real numbers. Let $y = A \sin(\omega t + \phi)$. Show that we can write

$$y = C_1 \cos \omega t + C_2 \sin \omega t$$

for some constants $C_1$ and $C_2$. Hint: use a trigonometric identity.

**Solution:** Recall the trigonometric identity $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$. Applying this identity with $\alpha = \omega t$ and $\beta = \phi$, we get

$$y = A \sin(\omega t + \phi) = A(\sin \omega t \cos \phi + \cos \omega t \sin \phi)$$

$$= A \sin \phi \cos \omega t + A \cos \phi \sin \omega t = C_1 \cos \omega t + C_2 \sin \omega t,$$

for $C_1 = A \sin \phi$ and $C_2 = A \cos \phi$.

b.) Suppose $C_1, C_2$, and $\omega$ are real numbers. Let $y = C_1 \cos \omega t + C_2 \sin \omega t$. Show that we can write

$$y = A \sin(\omega t + \phi)$$

for some real numbers $A$ and $\phi$, with $A \geq 0$. Hint: You can use the fact from trigonometry that any point on the unit circle (given by the equation $x^2 + y^2 = 1$) can be written in the form $(\cos \theta, \sin \theta)$ for some real number $\theta$.

**Solution:** We claim that we can find $\phi$ and $A \geq 0$ such that $A \sin \phi = C_1$ and $A \cos \phi = C_2$. Assuming the claim, then by part a., we have $A \sin(\omega t + \phi) = A \sin \phi \cos \omega t + A \cos \phi \sin \omega t = C_1 \cos \omega t + C_2 \sin \omega t$, as needed.

To prove the claim, we first see what condition $A$ would need to satisfy if the conditions $A \sin \phi = C_1$ and $A \cos \phi = C_2$ hold. If they hold, then

$$C_1^2 + C_2^2 = A^2 \sin^2 \phi + A^2 \cos^2 \phi = A^2 (\cos^2 \phi + \sin^2 \phi) = A^2,$$

by the trigonometric identity $\cos^2 \phi + \sin^2 \phi = 1$. Since we want $A \geq 0$, this means that there is no choice for $A$; we must have

$$A = \sqrt{C_1^2 + C_2^2}.$$

If $C_1 = C_2 = 0$ then $A = 0$ and we can pick any $\phi$, because $y = 0$ in this case. From now on, assume that $C_1$ and $C_2$ are not both 0, so that $A > 0$. To find $\phi$, we need to solve $\cos \phi = C_2/A$ and $\sin \phi = C_1/A$. By the remark in the hint, we can find such a $\phi$ if $(C_2/A)^2 + (C_1/A)^2 = 1$. This condition is the same as $C_1^2 + C_2^2 = A^2$, which holds
because this is how we have chosen \( A \). Therefore such a \( \phi \) exists. (To find \( \phi \) explicitly, note that \( \tan \phi = \frac{\sin \phi}{\cos \phi} = \frac{C_1/A}{C_2/A} = \frac{C_1}{C_2} \), so if \( C_1 > 0 \) we can take \( \phi = \tan^{-1}(C_1/C_2) \), but if \( C_1 \leq 0 \) we have to be careful about which quadrant \( \phi \) is in.)

2.) A spring has mass 2 kg and spring constant (stiffness) 32 N/m. Suppose that the spring is undamped. If it is stretched 2 m beyond its natural length and given an initial velocity of 12 m/sec in the same direction it is stretched, find the maximum displacement from equilibrium the spring will reach before changing direction back toward equilibrium. Hint: what you did in Additional Problem 1 b may be useful.

Solution: The general mass-spring equation is \( my'' + by' + ky = 0 \). Here \( b = 0 \) because the spring is undamped, and we are given \( m = 2 \) and \( k = 32 \). Hence we have \( 2y'' + 32y = 0 \), or, after dividing through by 2,

\[
y'' + 16y = 0.
\]

The characteristic equation is \( r^2 + 16 = 0 \). Hence \( r^2 = -16 \), or \( r = \pm 4i = 0 \pm 4i \). Hence the general equation of motion of the spring is

\[
y = C_1 e^{0t} \cos 4t + C_2 e^{0t} \sin 4t = C_1 \cos 4t + C_2 \sin 4t,
\]

for arbitrary constants \( C_1 \) and \( C_2 \). We are given that \( y(0) = 2 \). Hence

\[
2 = C_1 \cos 0 + C_2 \sin 0 = C_1 \cdot 1 + C_2 \cdot 0 = C_1.
\]

Hence \( y = 2 \cos 4t + C_2 \sin 4t \). Therefore

\[
y' = -8 \sin 4t + 4C_2 \cos 4t.
\]

We are given that \( y'(0) = 12 \). Hence

\[
12 = -8 \sin 0 + 4C_2 \cos 0 = -8 \cdot 0 + 4C_2 \cdot 1 = 4C_2.
\]

Therefore \( C_2 = 3 \). Hence

\[
y = 2 \cos 4t + 3 \sin 4t.
\]

By problem 1, we can write \( y = A \sin(4t + \phi) \), for some \( \phi \), where \( A = \sqrt{2^2 + 3^2} = \sqrt{13} \). That is,

\[
y = \sqrt{13} \sin(4t + \phi).
\]

The function \( \sin(4t + \phi) \) reaches a maximum of 1, and hence the maximum reached by \( y \) is \( \sqrt{13} \).

3.) A spring has mass 3 kg, damping constant 12 N-sec/m, and spring constant (stiffness) 9 N/m. If the spring is stretched 2 m and given an intial velocity of 12 m/sec in the direction opposite to the direction in which it was pulled, find the time the spring takes to first return to equilibrium position. Does the spring return to equilibrium position finitely many times (if so, how many) or infinitely many times?

Solution: The general mass-spring equation is \( my'' + by' + ky = 0 \). Here we are given \( m = 3, b = 12 \) and \( k = 9 \). Hence we have \( 3y'' + 12y' + 9y = 0 \), or, after dividing through by 3,

\[
y'' + 4y' + 3y = 0.
\]
The characteristic equation is \( r^2 + 4r + 3 = 0 \). The left side factors to give \((r + 3)(r + 1) = 0\). Hence the roots are \( r = -1 \) and \(-3\), and the general equation of motion of the spring is

\[
y = C_1 e^{-t} + C_2 e^{-3t},
\]

for arbitrary constants \( C_1 \) and \( C_2 \). Therefore

\[
y' = -C_1 e^{-t} - 3C_2 e^{-3t}
\]

We are given that \( y(0) = 2 \) and \( y'(0) = -12 \) (because the initial velocity is in the direction opposite to the stretching). From \( y(0) = 2 \) we get

\[
(1) \quad 2 = C_1 e^0 + C_2 e^0 = C_1 + C_2.
\]

From \( y'(0) = -12 \), we have

\[
(2) \quad -12 = -C_1 e^0 + -3C_2 e^0 = -C_1 - 3C_2.
\]

Adding equations (1) and (2), we obtain

\[
-10 = -2C_2,
\]

so \( C_2 = 5 \). From either equation, say from \( C_1 + C_2 = 2 \), we obtain \( C_1 = -3 \). Therefore

\[
y = -3e^{-t} + 5e^{-3t}.
\]

We want to know the first time after \( t = 0 \) when the spring returns to equilibrium position, that is, \( y = 0 \). So we set \( y = 0 \) and solve:

\[
0 = -3e^{-t} + 5e^{-3t}.
\]

Multiplying through by \( e^t \) gives \( 0 = -3 + 5e^{-2t} \), or \( 5e^{-2t} = 3 \), or \( e^{-2t} = 3/5 \). Taking the natural logarithm of both sides gives

\[
-2t = \ln \left( e^{-2t} \right) = \ln(3/5) = -\ln(5/3),
\]

hence

\[
t = \frac{1}{2} \ln \left( \frac{5}{3} \right) \approx .255 \text{ sec.}
\]

Since the equation \( y(t) = -3e^{-t} + 5e^{-3t} = 0 \) is equivalent to \( 5e^{-2t} = 3 \), it has only one solution (because \( 5e^{-2t} \) is a decreasing function of \( t \), so it only hits the value 3 once). So the spring only returns to equilibrium position once, on the way back from its initial stretch. After it passes through equilibrium, it reaches a minimum, then heads back toward equilibrium, this time approaching equilibrium asymptotically as \( t \to \infty \).

4.) Solve \( y'' - 2y' - 3y = 3e^{2t} \).

**Solution:** (a) We first find \( y_h \), the solution to the homogeneous equation \( y'' - 2y' - 3y = 0 \). The characteristic equation is \( r^2 - 2r - 3 = 0 \), which factors as \((r - 3)(r + 1) = 0\). Hence \( r = 3 \) or \( r = -1 \). Therefore

\[
y_h = C_1 e^{3t} + C_2 e^{-t},
\]
for arbitrary constants $C_1$ and $C_2$.

(b) Next we find a particular solution $y_p$ of $y'' - 2y' - 3y = 3e^{2t}$. The inhomogeneous part is $3e^{2t}$, which is not a constant multiple of either term in $y_h$, so we can take

$$y_p = Ae^{2t}.$$ 

Then $y'_p = 2 Ae^{2t}$ and $y''_p = 4 Ae^{2t}$. Hence

$$y''_p - 2y'_p - 3y_p = 4 Ae^{2t} - 4 Ae^{2t} - 3 Ae^{2t} = -3 Ae^{2t}.$$ 

Hence $y''_p - 2y'_p - 3y_p = 3e^{2t}$ will be satisfied if we choose $A = -1$. Therefore

$$y_p = -e^{2t}.$$ 

(c) Finally, the general solution of $y'' - 2y' - 3y = 3e^{2t}$ is $y = y_p + y_h$, or

$$y = -e^{2t} + C_1e^{3t} + C_2e^{-t},$$ 

for arbitrary constants $C_1$ and $C_2$.

5.) Solve $y'' + y' - 2y = 2t, \quad y(0) = 0, \quad y'(0) = 1$.

**Solution:** (a) We first find $y_h$, the solution to the homogeneous equation $y'' + y' - 2y = 0$. The characteristic equation is $r^2 + r - 2 = 0$, which factors as $(r+2)(r-1) = 0$. Hence $r = -2$ or $r = 1$. Therefore

$$y_h = C_1e^{-2t} + C_2e^t,$$ 

for arbitrary constants $C_1$ and $C_2$.

(b) Next we find a particular solution $y_p$ of $y'' + y' - 2y = 2t$. The inhomogeneous part is $2t$, which is not a constant multiple of either term in $y_h$. Since $2t$ is a polynomial of degree 1, we take

$$y_p = At + B,$$ 

the general polynomial of degree 1. Then $y'_p = A$ and $y''_p = 0$. Hence


Hence $y''_p + y'_p - 2y_p = 2t$ will be satisfied if choose $-2A = 2$ and $A - 2B = 0$. From $-2A = 2$ we get $A = -1$. Then $A - 2B = 0$ implies $B = -1/2$. Therefore

$$y_p = -t - \frac{1}{2}.$$ 

(c) Finally, the general solution of $y'' + y' - 2y = 2t$ is $y = y_p + y_h$, or

$$y = -t - \frac{1}{2} + C_1e^{-2t} + C_2e^t,$$ 

for arbitrary constants $C_1$ and $C_2$. 
(d) Then \( y' = -1 - 2C_1e^{-2t} + C_2e^t \). From \( y(0) = 0 \) we get
\[
0 = -0 - \frac{1}{2} + C_1e^0 + C_2e^0 = -\frac{1}{2} + C_1 + C_2,
\]
or
\[
(3) \quad C_1 + C_2 = \frac{1}{2}.\]
From \( y'(0) = 1 \), we get
\[
1 = -1 - 2C_1e^0 + C_2e^0 = -1 - 2C_1 + C_2,
\]
or
\[
(4) \quad -2C_1 + C_2 = 2.\]
Subtracting equation (4) and (3) gives \( 3C_1 = -3/2 \), hence \( C_1 = -1/2 \). Substituting \( C_1 = -1/2 \) and solving for \( C_2 \) in either equation gives \( C_2 = 1 \). Hence
\[
y = -t - \frac{1}{2} - \frac{1}{2}e^{-2t} + e^t.\]

6.) Solve \( y'' - 2y' - 3y = \cos 2t \).

**Solution:** (a) We first find \( y_h \), the solution to the homogeneous equation \( y'' - 2y' - 3y = 0 \). The characteristic equation is \( r^2 - 2r - 3 = 0 \), which factors as \( (r - 3)(r + 1) = 0 \). Hence \( r = 3 \) or \( r = -1 \). Therefore
\[
y_h = C_1e^{3t} + C_2e^{-t},\]
for arbitrary constants \( C_1 \) and \( C_2 \).

(b) Next we find a particular solution \( y_p \) of \( y'' - 2y' - 3y = \cos 2t \). The inhomogeneous part is \( \cos 2t \), which is not a constant multiple of either term in \( y_h \). We take
\[
y_p = A \cos 2t + B \sin 2t.\]
Then \( y_p' = -2A \sin 2t + 2B \cos 2t \) and \( y_p'' = -4A \cos 2t - 4B \sin 2t \). Hence
\[
y_p'' - 2y_p' - 3y_p = -4A \cos 2t - 4B \sin 2t - 2(-2A \sin 2t + 2B \cos 2t) - 3(A \cos 2t + B \sin 2t) = -4A \cos 2t - 4B \sin 2t + 4A \sin 2t - 4B \cos 2t - 3A \cos 2t - 3B \sin 2t = (-4A - 4B - 3A) \cos 2t + (-4B - 4A - 3B) \sin 2t = (-7A - 4B) \cos 2t + (4A - 7B) \sin 2t.\]
Hence \( y_p'' - 2y_p' - 3y_p = \cos 2t \) will be satisfied if choose \(-7A - 4B = 1 \) and \( 4A - 7B = 0 \).
We multiply the first equation by \( 4 \) to get \(-28A - 16B = 4 \), and we multiply the second equation by \( 7 \) to get \( 28A - 49B = 0 \). Adding the results gives \(-65B = 4 \), hence \( B = -4/65 \). Using \( 4A - 7B = 0 \) gives \( A = 7B/4 = -7/65 \). Therefore
\[
y_p = -\frac{7}{65} \cos 2t - \frac{4}{65} \sin 2t.\]

(c) Finally, the general solution of \( y'' - 2y' - 3y = \cos 2t \) is \( y = y_p + y_h \), or
\[
y = -\frac{7}{65} \cos 2t - \frac{4}{65} \sin 2t + C_1e^{3t} + C_2e^{-t},\]
for arbitrary constants \( C_1 \) and \( C_2 \).
7.) Solve \( y'' + 5y' + 4y = 6e^{-t} \).

**Solution:** (a) We first find \( y_h \), the solution to the homogeneous equation \( y'' + 5y' + 4y = 0 \). The characteristic equation is \( r^2 + 5r + 4 = 0 \), which factors as \((r + 4)(r + 1) = 0\). Hence \( r = -4 \) or \( r = -1 \). Therefore

\[
y_h = C_1 e^{-4t} + C_2 e^{-t},
\]

for arbitrary constants \( C_1 \) and \( C_2 \).

(b) Next we find a particular solution \( y_p \) of \( y'' + 5y' + 4y = 6e^{-t} \). The inhomogeneous part is \( 6e^{-t} \), which is a constant multiple of the term \( C_2 e^{-t} \) in \( y_h \). Hence we replace our usual guess \( Ae^{-t} \) by the modified guess

\[
y_p = Ae^{-t}.
\]

Then by the product rule,

\[
y'_p = Ae^{-t} - Ae^{-t}
\]

and

\[
y''_p = -Ae^{-t} - Ae^{-t} + Ae^{-t} = -2Ae^{-t} + Ae^{-t}.
\]

Hence

\[
y''_p + 5y'_p + 4y_p = -2Ae^{-t} + Ae^{-t} + 5(Ae^{-t} - Ae^{-t}) + 4Ae^{-t}
\]
\[
= -2Ae^{-t} + Ae^{-t} + 5Ae^{-t} - 5Ae^{-t} + 4Ae^{-t}
\]
\[
= (-2A + 5A)e^{-t} + (A - 5A + 4A)te^{-t} = 3Ae^{-t} + 0 = 3Ae^{-t}.
\]

Hence \( y''_p + 5y'_p + 4y_p = 6e^{-t} \) will be satisfied if choose \( 3A = 6 \), or \( A = 2 \). Therefore

\[
y_p = 2te^{-t}.
\]

(c) Finally, the general solution of \( y'' + 5y' + 4y = 6e^{-t} \) is \( y = y_p + y_h \), or

\[
y = 2te^{-t} + C_1 e^{-4t} + C_2 e^{-t},
\]

for arbitrary constants \( C_1 \) and \( C_2 \).