Math 231.04, Problem Set 6 Solutions (Partial)

Due Wednesday, Feb. 24, 2010

From Text Fundamentals of Differential Equations, by Nagle, Saff, and Snider

Section 4.1, # 3, 4, 5
Section 4.2, # 3, 5, 7, 11, 13, 15, 39, 41

Additional Problems:

1.) Which of the following functions are solutions of the differential equation

\[ y \cdot y'' + (y')^2 = 0 \]

(a) \( y = 1 \);
(b) \( y = \sqrt{x} \);
(c) \( y = 1 + \sqrt{x} \).

What does this problem say about the principle of superposition?

**Solution:** (a) For \( y = 1 \), we have \( y' = 0 \) and \( y'' = 0 \). Hence

\[ y \cdot y'' + (y')^2 = 1 \cdot 0 + 0^2 = 0 + 0 = 0, \]

so \( y = 1 \) is a solution.

(b): For \( y = \sqrt{x} = x^{1/2} \), we have \( y' = \frac{1}{2}x^{-1/2} \) and \( y'' = -\frac{1}{4}x^{-3/2} \). Hence

\[ y \cdot y'' + (y')^2 = x^{1/2} \cdot \left( -\frac{1}{4} \right) x^{-3/2} + \left( \frac{1}{2}x^{-1/2} \right)^2 = -\frac{1}{4}x^{-1} + \frac{1}{4}x^{-1} = 0, \]

so \( y = x^{1/2} \) is a solution.

(c): For \( y = 1 + \sqrt{x} = 1 + x^{1/2} \), we have \( y' = \frac{1}{2}x^{-1/2} \) and \( y'' = -\frac{1}{4}x^{-3/2} \). Hence

\[ y \cdot y'' + (y')^2 = \left( 1 + x^{1/2} \right) \cdot \left( -\frac{1}{4} \right) x^{-3/2} + \left( \frac{1}{2}x^{-1/2} \right)^2 \]

\[ = -\frac{1}{4}x^{-3/2} - \frac{1}{4}x^{-1} + \frac{1}{4}x^{-1} = -\frac{1}{4}x^{-3/2}, \]

which is not 0. Hence \( y = 1 + \sqrt{x} \) is not a solution.

The principle of superposition does not hold for the equation \( y \cdot y'' + (y')^2 = 0 \), because \( y = 1 \) and \( y = \sqrt{x} \) are solutions but \( y = 1 + \sqrt{x} \) is not a solution. The reason the principle of superposition does not apply to this equation is that it is not linear. Thus, this problem shows that the principle of superposition does not apply to nonlinear equations in general.

2.) Solve \( y'' + 2y' - 15y = 0 \).
Solution: We try \( y = e^{rt} \). Then \( y' = re^{rt} \) and \( y'' = r^2e^{rt} \). Substituting into the equation, we get
\[
r^2e^{rt} + 2re^{rt} - 15e^{rt} = 0.
\]
Factoring out the term \( e^{rt} \) gives
\[
e^{rt} \left( r^2 + 2r - 15 \right) = 0.
\]
The product can only be 0 if one of the terms is 0, and \( e^{rt} \) is never 0. Hence we only have a solution if the characteristic equation
\[
r^2 + 2r - 15 = 0
\]
holds. (In homework or on exams, you can jump right to the characteristic equation, you don’t have to go through all the steps above.) Factoring, this gives \((r+5)(r-3) = 0\), so the solutions are \( r = -5 \) and \( r = 3 \). Hence \( y_1 = e^{-5t} \) and \( y_2 = e^{3t} \) are solutions. Any linear combination of \( y_1 \) and \( y_2 \) is a solution (by the principle of superposition). So for any constants \( C_1 \) and \( C_2 \),
\[
y = C_1e^{-5t} + C_2e^{3t}
\]
is a solution, and, as we show in class, every solution is of this form.

3.) Solve \( y'' - 4y' + 3y = 0, \ y(0) = 7, \ y'(0) = 11 \).

Solution: The characteristic equation (see the solution to problem for the steps and their justification) is
\[
r^2 - 4r + 3 = 0.
\]
Factoring, this gives \((r - 1)(r - 3) = 0\), so \( r = 1 \) or \( r = 3 \). Hence the general solution is
\[
y = C_1e^t + C_2e^{3t}.
\]
Then
\[
y' = C_1e^t + 3C_2e^{3t}.
\]
The condition \( y(0) = 7 \) gives, by setting \( t = 0 \) and \( y = 7 \) in (1),
\[
C_1 + C_2 = 7
\]
(since \( e^0 = 1 \)). The condition \( y'(0) = 11 \) gives, by setting \( t = 0 \) and \( y' = 11 \) in (2),
\[
C_1 + 3C_2 = 11.
\]
We need to solve (3) and (4) simultaneously. The easiest was is to multiply equation (3) by \(-1\) to get
\[
-C_1 - C_2 = -7
\]
and add this equation to (4) to get
\[
2C_2 = 4,
\]
hence $C_2 = 2$. Then we can substitute $C_2 = 2$ into either (3) or (4) to get $C_1 = 5$. The general idea is to multiply one equation by whatever number you need, so that when it is added to the second equation, one of the unknowns cancels out. Here we obtain the answer

$$y = 5e^t + 2e^{3t}.$$

4.) Solve $y'' + 6y' + 4y = 0$.

**Solution:** The characteristic equation is

$$r^2 + 6r + 4 = 0.$$

This doesn’t factor nicely, so we use the quadratic formula: the solutions of $ax^2 + bx + c = 0$ are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Here $a = 1$, $b = 6$, and $c = 4$. Hence

$$r = \frac{-6 \pm \sqrt{36 - 4(1)(4)}}{2} = \frac{-6 \pm \sqrt{20}}{2} = -3 \pm \sqrt{5}.$$

Hence $y_1 = e^{(-3+\sqrt{5})t}$ and $y_2 = e^{(-3-\sqrt{5})t}$ are solutions. Hence so is

$$y = C_1 e^{(-3+\sqrt{5})t} + C_2 e^{(-3-\sqrt{5})t},$$

for any constants $C_1$ and $C_2$, and every solution is of this form.

5.) Solve $y'' + 4y' + 4y = 0$, $y(0) = 2$, $y'(0) = -1$.

**Solution:** The characteristic equation is

$$r^2 + 4r + 4 = 0,$$

which factors as $(r + 2)^2 = 0$. Hence $r = -2$ is a repeated root. Hence $y_1 = e^{-2t}$ is a solution, and because of the repeated root, so is $y_2 = te^{-2t}$. Therefore the general solution is

$$(5) \quad y = C_1 e^{-2t} + C_2 te^{-2t}. $$

Hence, using the product rule to take the derivative of the second term,

$$(6) \quad y' = -2C_1 e^{-2t} + C_2 e^{-2t} - 2C_2 te^{-2t}.$$  

Using (5), the condition $y(0) = 2$ gives

$$2 = C_1 \cdot e^0 + C_2 \cdot 0 \cdot e^0 = C_1 \cdot 1 = C_1.$$  

Also, (6) and the condition $y'(0) = -1$ gives

$$-1 = -2C_1 e^0 + C_2 e^0 - 2C_2 \cdot 0 \cdot e^0 = -2C_1 + C_2.$$  

Since $C_1 = 2$, we have $-1 = -4 + C_2$, so $C_2 = 3$. Hence, substituting into (5),

$$y = 2e^{-2t} + 3te^{-2t}.$$
6.) Solve $y''' - y'' - 4y' + 4y = 0$.

**Solution:** The characteristic equation is
\[ r^3 - r^2 - 4r + 4 = 0. \]

To find a factor, we first try to obtain a root of $p(r) = r^3 - r^2 - 4r + 4$. After some guessing, we see that
\[ p(1) = 1^3 - 1^2 - 4 \cdot 1 + 4 = 0. \]

By the root/factor theorem, we deduce that $r - 1$ divides evenly into $r^3 - r^2 - 4r + 4$. Dividing using long division of polynomials gives
\[ r^3 - r^2 - 4r + 4 = (r - 1)(r^2 - 4) = (r - 1)(r + 2)(r - 2). \]

Hence the roots are $r = 1$, $r = -2$, and $r = 2$. Therefore $y_1 = e^t$, $y_2 = e^{-2t}$, and $y_3 = e^{2t}$ are solutions. By the superposition principle,
\[ y = C_1 e^t + C_2 e^{-2t} + C_3 e^{2t} \]
is a solution, for any constants $C_1$, $C_2$, and $C_3$. By considerations discussed in class, $y$ is the general solution.