1.) (20 points) Solve
\[ y'' - 4y' + 5y = 0, \quad y(0) = 3, \quad y'(0) = 4. \]

We guess a solution of the form \( y = e^{rt} \), which leads to the characteristic equation
\[ r^2 - 4r + 5 = 0. \]

We cannot factor this quadratic, so using the quadratic formula we get
\[ r = \frac{4 \pm \sqrt{16 - 4 \cdot 5}}{2} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm \sqrt{4} \sqrt{-1}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i. \]

This implies that the general solution is
\[ y = C_1 e^{2t} \cos t + C_2 e^{2t} \sin t. \]

Then the condition \( y(0) = 3 \) yields
\[ 3 = C_1 e^0 \cos 0 + C_2 e^0 \sin 0 = C_1 + 0 = C_1. \]

Hence
\[ y = 3e^{2t} \cos t + C_2 e^{2t} \sin t. \]

Next, we calculate \( y' \) using the product rule:
\[ y' = 6e^{2t} \cos t - 3e^{2t} \sin t + C_2 2e^{2t} \sin t + C_2 e^{2t} \cos t. \]

Hence the condition \( y'(0) = 4 \) yields
\[ 4 = 6 - 0 + 0 + C_2. \]

Hence \( C_2 = 4 - 6 = -2 \). So finally
\[ y = 3e^{2t} \cos t - 2e^{2t} \sin t. \]
2.) (20 points) Assuming the fact that the function \( y_1 = e^t \) is a solution of the equation

\[
(*) \quad y'' - \left( 2 + \frac{2}{t} \right) y' + \left( 1 + \frac{2}{t} \right) y = 0,
\]

find the general solution of (*)

To get the general solution, we need two linearly independent solutions. Since we have one solution, we use the method of reduction of order. We guess a second solution of the form \( y_2 = vy_1 \), where \( v \) is a function of \( t \). Here \( y_2 = ve^t = e^tv \); hence by the product rule, \( y_2' = e^tv + e^t v' \) and \( y_2'' = e^tv + e^tv' + e^tv' + e^t v'' = e^tv + 2e^tv' + e^t v'' \). Substituting this in the equation, we get

\[
y_2'' - \left( 2 + \frac{2}{t} \right) y_2' + \left( 1 + \frac{2}{t} \right) y_2 = e^tv + 2e^tv' + e^tv'' - \left( 2 + \frac{2}{t} \right) (e^tv + e^t v') + \left( 1 + \frac{2}{t} \right) e^tv
\]

\[
= e^t v + 2e^t v' + e^t v'' - 2e^t v - 2e^t v' - \frac{2}{t} e^t v + e^t v + \frac{2}{t} e^t v
\]

\[
= e^t v'' + \left( 2e^t - 2e^t - \frac{2}{t} e^t \right) v' + \left( e^t - 2e^t - \frac{2}{t} e^t + e^t + \frac{2}{t} e^t \right) v
\]

\[
= e^t v'' - \frac{2}{t} e^t v'.
\]

Setting this equal to 0 and we get the equation \( e^t v'' - \frac{2}{t} e^t v' = 0 \). Dividing through by \( e^t \) we get

\[
v'' - \frac{2}{t} v' = 0.
\]

We make the substitution \( u = v' \). This gives the equation \( u' - \frac{2}{t} u = 0 \), which is a first order linear equation which we can solve by the method of the integrating factor. The integrating factor is

\[
e^\int -\frac{2}{t} \, dt = e^{-2 \ln t} = e^{\ln (t^{-2})} = t^{-2} = \frac{1}{t^2}.
\]

Multiplying through the equation \( u' - \frac{2}{t} u = 0 \) by the integrating factor \( \frac{1}{t^2} \) gives

\[
\frac{1}{t^2} u' - \frac{2}{t^3} u = 0.
\]

The left side is \( \left( \frac{1}{t^2} u \right)' \), by the product rule. So we have \( \left( \frac{1}{t^2} u \right)' = 0 \), which implies \( \frac{1}{t^2} u = C_1 \), or \( u = C_1 t^2 \). Then since \( u = v' \), we get \( v = \int u = \int C_1 t^2 \, dt = C_1 t^3/3 + C_2 \). Then

\[
y_2 = ve^t = \left( C_1 \frac{t^3}{3} + C_2 \right) e^t = \frac{C_1}{3} t^3 e^t + C_2 e^t.
\]

Since we are just looking for one \( y_2 \), we can take \( C_1 = 3 \) and \( C_2 = 0 \) to obtain \( y_2 = t^3 e^t \). Then the general solution is \( y = C_1 y_1 + C_2 y_2 \), or

\[
y = C_1 e^t + C_2 t^3 e^t,
\]

for arbitrary constants \( C_1 \) and \( C_2 \).
3.) (10 points) For the equations below, write down the general form of the particular solution $y_p$ that you would use in the method of undetermined coefficients. Just write down the form; do not attempt to solve for the coefficients.

a.) (5 points) $y'' - 9y = t^2 \sin 4t$.

We first find $y_h$, the solution to the homogeneous equation $y'' - 9y = 0$. For this we guess $y_h = e^{rt}$, which leads to the characteristic equation $r^2 - 9 = 0$. Factoring, we get $(r + 3)(r - 3) = 0$, so $r_1 = -3$ and $r_2 = +3$ are the roots of the characteristic equation. Thus the general form of $y_h$ is $y_h = C_1 e^{-3t} + C_2 e^{3t}$.

Now we look for a particular solution $y_p$. The right side of the equation is $t^2 \sin 4t$ we guess $y_p = (At^2 + Bt + C) \sin 4t + (Dt^2 + Et + F) \cos 4t$. Since none of these terms, after you multiply them out, has any term in which is a constant multiple of any term in $y_h$, this form of $y_p$ is sufficient.

Hence $y_p = (At^2 + Bt + C) \sin 4t + (Dt^2 + Et + F) \cos 4t$.

b.) (5 points) $y'' - 9y = te^{3t}$.

The left side of the equation is the same as for part a., so we have the same homogeneous solution $y_h = C_1 e^{-3t} + C_2 e^{3t}$. Now we look for a particular solution $y_p$. The right side of the equation is of the form $te^{3t}$ we try $y_p = (At + B)e^{3t} = Ae^{3t} + Be^{3t}$. However, the term $Ae^{3t}$ is a constant multiple of the term $C_2 e^{3t}$ in $y_h$, so we need to modify $y_p$ by multiplying by $t$ to obtain $y_p = t(At + B)e^{3t} = At^2 e^{3t} + Bte^{3t}$. This no longer has any overlap with $y_h$, so this form of $y_p$ is sufficient. So $y_p = At^2 e^{3t} + Bte^{3t}$.

4.) (10 points) Solve the Cauchy-Euler equation $t^2 y'' + 7ty' + 9y = 0$, for $t > 0$.

Because this is a Cauchy-Euler equation, we try $y = t^r$ as a solution. Then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Substituting into the equation gives

$$t^2 r(r-1)t^{r-2} + 7rt^{r-1} + 9t^r = 0,$$

or

$$r(r-1)t^r + 7rt^r + 9t^r = 0.$$  

Cancelling $t^r$ gives $r(r-1) + 7r + 9 = 0$, or

$$0 = r^2 + 6r + 9 = (r + 3)^2.$$  

Hence $r = -3$ is a double root, and we get solutions $y_1 = t^{-3}$ and $y_2 = t^{-3} \ln t$. The general solution $y$ is a linear combination of $y_1$ and $y_2$; that is,

$$y = C_1 t^{-3} + C_2 t^{-3} \ln t,$$

for arbitrary constants $C_1$ and $C_2$. 

5.) (a) (5 points) Check that the functions $y_1 = t$ and $y_2 = t^3$ are solutions of

$$y'' - \frac{3}{t} y' + \frac{3}{t^2} y = 0, \quad t > 0.$$ 

Since $y_1 = t$, we get $y_1' = 1$ and $y_1'' = 0$. Therefore

$$y_1'' - \frac{3}{t} y_1' + \frac{3}{t^2} y_1 = 0 - \frac{3}{t} + \frac{3}{t^2} \cdot t = \frac{3}{t} - \frac{3}{t} = 0,$$

hence $y_1$ is a solution.

For $y_2 = t^3$ we get $y_2' = 3t^2$ and $y_2'' = 6t$. Hence

$$y_2'' - \frac{3}{t} y_2' + \frac{3}{t^2} y_2 = 6t - \frac{3}{t} 3t^2 + \frac{3}{t^2} t^3 = 6t - 9t + 3t = 0,$$

so $y_2$ is a solution.

(b) (15 points) Find the general solution of

$$y'' - \frac{3}{t} y' + \frac{3}{t^2} y = t^2, \quad t > 0.$$ 

To find the general solution, we need to find a particular solution $y_p$. Using the method of variation of parameters, we guess a solution of the form $y_p = v_1 y_1 + v_2 y_2$. The equations determining $v'_1$ and $v'_2$ are $y_1 v'_1 + y_2 v'_2 = 0$ and $y'_1 v'_1 + y'_2 v'_2 = g(t)$, where $g(t)$ is the right side of the equation. Using our values of $y_1, y_2$ and $g$, we get

$$t v'_1 + t^3 v'_2 = 0$$
$$v'_1 + 3 t^2 v'_2 = t^2.$$ 

To solve these equations simultaneously for $v'_1$ and $v'_2$, it is easiest to first cancel a factor of $t$ from the first equation to get $v'_1 + t^2 v'_2 = 0$. Then when we subtract this equation from the second equation $v'_1 + 3 t^2 v'_2 = t^2$ to get $2 t^2 v'_2 = t^2$. Hence $v'_2 = 1/2$. Substituting $v'_2 = 1/2$ into $v'_1 + t^2 v'_2 = 0$ gives $v'_1 + \frac{1}{2} t^2 = 0$ or $v'_1 = -\frac{1}{2} t^2$.

Integrating the equation $v'_1 = -\frac{1}{2} t^2$ gives $v_1 = -\frac{1}{2} \frac{t^3}{3} = -\frac{1}{6} t^3$ while integrating $v'_2 = 1/2$ gives $v_2 = \frac{1}{2} t$ (the constants of integration are not needed here). Hence

$$y_p = v_1 y_1 + v_2 y_2 = -\frac{1}{6} t^3 \cdot t + \frac{1}{2} t \cdot t^3 = \left( -\frac{1}{6} + \frac{1}{2} \right) t^4 = \left( -\frac{1}{6} + \frac{3}{6} \right) t^4 = \frac{2}{6} t^4 = \frac{1}{3} t^4.$$

Then the general solution is $y = y_h + y_p$, or

$$y = C_1 t + C_2 t^3 + \frac{1}{3} t^4,$$

for arbitrary constants $C_1, C_2$, and $C_3$. 
6.) (10 points) A spring has mass 2 kg, damping constant 8 N-sec/m, and spring constant (stiffness) 26 N/m. There is no external force driving the spring. Find the general formula for the displacement from equilibrium position of the spring at time \( t \) seconds, and determine whether this spring is undamped, underdamped, critically damped, or overdamped.

The general equation for the displacement from equilibrium \( y \) of the spring is \( my'' + by' + ky = 0 \), where \( m \) is the mass, \( b \) is the stiffness, and \( k \) is the spring constant. Here we get \( 2y'' + 8y' + 26y = 0 \), or, after dividing by 2,

\[
y'' + 4y' + 13y = 0.
\]

The characteristic equation is \( r^2 + 4r + 13 = 0 \). We cannot factor this quadratic, so using the quadratic formula we get

\[
r = \frac{-4 \pm \sqrt{16 - 4 \cdot 13}}{2} = \frac{-4 \pm \sqrt{-36}}{2} = \frac{-4 \pm 6i}{2} = -2 \pm 3i.
\]

This implies that the general solution is

\[
y = C_1 e^{-2t} \cos 3t + C_2 e^{-2t} \sin 3t,
\]

for arbitrary constants \( C_1 \) and \( C_2 \). This spring is underdamped.

7.) (10 points) Solve \( y''' - 4y'' = 48t \).

We first find \( y_h \), the solution to the homogeneous equation \( y''' - 4y'' = 0 \). The characteristic equation is \( r^3 - 4r^2 = 0 \), or \( r^2(r - 4) = 0 \). Hence \( r = 4 \) is a root and \( r = 0 \) is a double root. Therefore

\[
y_h = C_1 e^{4t} + C_2 e^{0t} + C_3 t e^{0t} = C_1 e^{4t} + C_2 + C_3 t,
\]

for arbitrary constants \( C_1, C_2 \) and \( C_3 \).

Next we find a particular solution \( y_p \) of \( y''' - 4y'' = 48t \). Because the inhomogeneous part on the right hand side of the equation is \( 48t \), a polynomial of degree 1, the usual guess for \( y_p \) is \( At + B \), the general polynomial of degree 1. However, \( At \) is a constant multiple of the term \( C_3 t \) in \( y_h \) (and \( B \) is a multiple of the term \( C_2 \)), so we multiply by \( t \) and try \( t(At + B) = At^2 + Bt \). However, \( Bt \) is a constant multiple of the term \( C_3 t \) in \( y_h \), so we multiply by \( t \) again, to arrive at

\[
y_p = t(At^2 + B) = At^3 + Bt^2.
\]

This is not a multiple of any term in \( y_h \), so it should work. We calculate

\[
y_p' = 3At^2 + 2Bt, \quad y_p'' = 6At + 2B, \quad \text{and} \quad y_p''' = 6A.
\]

Hence

\[
y_p''' - 4y_p'' = 6A - 4(6At + 2B) = 6A - 24At - 8B = -24At + 6A - 8B.
\]

Therefore \( y''' - 4y'' = 48t \) holds if \(-24A = 48 \) and \( 6A - 8B = 0 \). Therefore \( A = -2 \) and hence \( 8B = 6A = 6(-2) = -12 \), or \( B = -12/8 = -3/2 \). Therefore

\[
y_p = -2t^3 - \frac{3}{2} t^2.
\]

(c) Finally, the general solution of \( y''' - 4y'' = 48t \) is \( y = y_p + y_h \), or

\[
y = -2t^3 - \frac{3}{2} t^2 + C_1 e^{4t} + C_2 + C_3 t,
\]

for arbitrary constants \( C_1, C_2 \) and \( C_3 \).