1.) (20 points) A certain population of skunks has a birth rate (number of births per skunk per month) proportional to $\frac{1}{P^2}$, where $P = P(t)$ is the population at time $t$. Suppose their death rate is 0. Suppose there are originally 10 skunks, and at time 30 months there are 20 skunks. At what time, in months, will there be 40 skunks?

We are given that the birth rate $b$ is proportional to $\frac{1}{P^2}$, hence $b = \frac{k}{P^2}$ for some constant $k$. The death rate $d$ is 0. Hence $P'$, the rate of change of the population of skunks, is given by

$$\frac{dP}{dt} = P' = (b - d)P = \left( \frac{k}{P^2} - 0 \right) P = \frac{k}{P}.$$

The equation $\frac{dP}{dt} = \frac{k}{P}$ is separable, and we get $P \, dP = k \, dt$, hence $\int P \, dP = \int k \, dt$. Integrating each side and adding the constant of integration on the right side, we get

$$\frac{P^2}{2} = kt + C.$$

Multiplying by 2 and letting $k_1 = 2k$ and $C_1 = 2C$, we have $P^2 = 2kt + 2C = k_1 t + C_1$, or $P = \sqrt{k_1 t + C_1}$.

(We don’t need ± in front of the square root because the population can’t be negative.) Since the initial population of skunks is 10, we have $P(0) = 10$, so

$$10 = P(0) = \sqrt{k_1 \cdot 0 + C_1} = \sqrt{C_1},$$

so $C_1 = 100$. Hence we have $P = \sqrt{k_1 t + 100}$. Using this equation and the fact we are given that there are 20 skunks in 30 months, or $P(30) = 20$, we have

$$20 = \sqrt{k_1 \cdot 30 + 100}.$$

Squaring both sides gives $400 = 30k_1 + 100$, or $300 = 30k_1$, or $k_1 = 10$. Substituting in the general equation for $P$ above, we end up with

$$P = \sqrt{10t + 100}.$$

We want to know when there will be 40 skunks; that is, we want to find $t$ so that $P(t) = 40$. So we take our expression for $P(t)$ and set it equal to 40 and solve for $t$:

$$40 = \sqrt{10t + 100}.$$

Squaring both sides, we have $1600 = 10t + 100$, or $1500 = 10t$, or $t = 150$ months.
2.) A tank originally holds 20 gallons of water with a salt concentration of .2 lbs/gal. Salt water with a concentration of .1 lbs/gal is being pumped into the tank at a rate of 4 gal/hr while the (perfectly mixed) solution is being pumped out of the tank at the same rate.

(a) (15 points) Find the amount of salt in the tank \( t \) hrs after the start.

Let \( A(t) \) be the amount of salt in the tank at time \( t \). Then \( A(0) = (20 \text{ gal}) (.2 \text{ lbs/gal}) = 4 \text{ lbs} \).

Salt is coming into the tank at a rate of \( (4 \text{ gal/hr}) (.1 \text{ lbs/gal}) = .4 \text{ lbs/hr} \), while salt is leaving the tank at a rate of \( (4 \text{ gal/hr}) \left( \frac{A(t) \text{ lbs}}{20 \text{ gals}} \right) = \frac{2}{20} A(t) \frac{\text{ lbs}}{\text{ hr}} \), since the concentration of salt in the tank at time \( t \) is \( \frac{A(t) \text{ lbs}}{20 \text{ gals}} \). Hence

\[
A'(t) = .4 - .2A(t),
\]
or

\[
A'(t) + .2A(t) = .4.
\]

This is a first order linear differential equation, with integrating factor \( e^{\int .2 \, dt} = e^{.2t} \). Multiplying the equation through by \( e^{.2t} \) gives

\[
e^{.2t}A'(t) + .2e^{.2t}A(t) = .4e^{.2t}.
\]

Hence

\[
(e^{.2t}A)' = .4e^{.2t}.
\]

Integrating, we get

\[
e^{.2t}A = \frac{.4}{.2} e^{.2t} + C = 2e^{.2t} + C.
\]

Multiplying through by \( e^{-2t} \) gives

\[
A = 2 + Ce^{-2t}.
\]

To find \( C \), use \( A(0) = 4 \) to get \( 4 = 2 + C \). Hence \( C = 2 \) and we get

\[
A(t) = 2 + 2e^{-2t} \text{ lbs}.
\]

(b) (5 points) Intuitively, the limiting concentration (the limit of the concentration as \( t \to +\infty \)) should be .1 lbs/gal. Use your answer to part (a) to find the limiting concentration, and see if it agrees with this intuition.

Let \( C(t) \) be the concentration at time \( t \). Then \( C(t) = A(t)/20 \), since the volume is 20. So

\[
C(t) = \frac{1}{20} \left( 2 + 2e^{-2t} \right) = .1 + .1e^{-2t}.
\]

As \( t \to \infty \), we get \( e^{-2t} \to 0 \), hence \( .1 + .1e^{-2t} \to .1 \text{ lbs/gal} \) as \( t \to \infty \). So the limiting concentration is .1 lbs/gal, as we expect.
3.) (15 points) A rock of mass 2 kg is dropped from a cliff 300 feet high. Assume that the force, in newtons, due to air resistance is \(-4v\), where \(v\) is the velocity of the rock. The acceleration due to gravity is \(g = 9.8 \, \text{m/sec}^2\). Find the velocity, in \(\text{m/sec}\), of the rock 2 seconds after it is dropped.

We let \(t\) denote the time, in seconds, after the rock is dropped. By Newton’s law, we have \(F = ma = mv'\), where \(F\) is the net force acting on the rock. The force due to gravity is \(mg\), and the force due to air resistance is given as \(-4v\). Hence

\[
mv' = mg - 4v.
\]

Substituting \(m = 2\) and \(g = 9.8\) gives

\[
2v' = 2(9.8) - 4v.
\]

Dividing by 2, we have

\[
v' = 9.8 - 2v.
\]

We rewrite this equation as

\[
v' + 2v = 9.8.
\]

This equation is first-order and linear, with integrating factor \(e^{\int 2\, dt} = e^{2t}\). Multiplying the equation through by \(e^{2t}\) gives

\[
e^{2t}v' + 2e^{2t}v = 9.8e^{2t}.
\]

The left side is the derivative, by the product rule, of \(e^{2t}v\), so we have

\[
dt (e^{2t}v) = 9.8e^{2t}.
\]

Integrating both sides, we get

\[
e^{2t}v = \int dt (e^{2t}v) = \int 9.8e^{2t}\, dt = 4.9e^{2t} + C.
\]

Hence, multiplying the equation through by \(e^{-2t}\),

\[
v = 4.9 + Ce^{-2t}.
\]

At time 0, the rock is released, and hence has velocity 0. That is, \(v(0) = 0\). Substituting into the last equation for \(v\) gives

\[
0 = 4.9 + Ce^{0} = 4.9 + C,
\]

so \(C = -4.9\). Hence we obtain

\[
v = 4.9 - 4.9e^{-2t}.
\]

Hence the velocity at time \(t = 2\) is

\[
v(2) = 4.9 - 4.9e^{-2 \cdot 2} = 4.9 - 4.9e^{-4} = 4.9 - \frac{4.9}{e^4}.
\]

(b) (5 points) Find the distance the rock travels in the first 2 seconds after it is dropped.

To find the distance, we integrate the velocity function to find the position \(x = x(t)\):

\[
x = \int v(t)\, dt = \int 4.9 - 4.9e^{-2t}\, dt = 4.9t + 2.45e^{-2t} + C.
\]

If we take our coordinate system with origin at the point where the rock is released, then \(x(0) = 0\), so

\[
0 = 4.9 \cdot 0 + 2.45e^{0} + C = 2.45 + C,
\]

hence \(C = -2.45\). Therefore

\[
x = 4.9t + 2.45e^{-2t} - 2.45.
\]

Finally, the distance the rock falls by time \(t = 2\) seconds is

\[
x(2) = 4.9 \cdot 2 + 2.45e^{-2 \cdot 2} - 2.45 = 9.8 + 2.45e^{-4} - 2.45.
\]
4.) (10 points) Solve \(y'' + 2y' - 8y = 0, \ y(0) = 2, \ y'(0) = 22.\)

The characteristic equation is \(r^2 + 2r - 8 = 0.\) Factoring, we have \((r + 4)(r - 2) = 0,\) so \(r = -4\) or \(r = 2.\) Hence \(y_1 = e^{-4t}\) and \(y_2 = e^{2t}\) are solutions. Therefore the general solution is

\[y = C_1e^{-4t} + C_2e^{2t},\]

for constants \(C_1\) and \(C_2.\) Then

\[y' = -4C_1e^{-4t} + 2C_2e^{2t}.\]

The condition \(y(0) = 2\) gives, by substituting \(t = 0\) and \(y = 2\) into the equation for \(y,\)

\[C_1 + C_2 = 2.\]

The condition \(y'(0) = 22\) gives, by substituting into the equation for \(y',\)

\[-4C_1 + 2C_2 = 22.\]

Multiplying the equation \(C_1 + C_2 = 2\) by \(-2\) gives \(-2C_1 - 2C_2 = -4.\) Adding this result to the equation \(-4C_1 + 2C_2 = 22,\) the terms involving \(C_2\) cancel out, giving \(-6C_1 = 18,\) or \(C_1 = -3.\) Substituting \(C_1 = -3\) into \(C_1 + C_2 = 2\) gives \(C_2 = 5.\) Hence

\[y = -3e^{-4t} + 5e^{2t}.\]

5.) (10 points) Solve \(y'' + 10y' + 25y = 0, \ y(0) = 4, \ y'(0) = -5.\)

The characteristic equation is \(r^2 + 10r + 25 = 0,\) or \((r + 5)^2 = 0.\) Hence there is only one root, \(r = -5,\) which is repeated. Therefore \(y_1 = e^{-5t}\) and \(y_2 = te^{-5t}\) are solutions, and the general solution is

\[y = C_1e^{-5t} + C_2te^{-5t},\]

for constants \(C_1\) and \(C_2.\) Then, applying the product rule for the second term,

\[y' = -5C_1e^{-5t} + C_2e^{-5t} - 5C_2te^{-5t}.\]

The condition \(y(0) = 4\) gives, by substituting \(t = 0\) and \(y = 4\) into the equation for \(y,\)

\[C_1 + C_2 \cdot 0 = 4,\]

so \(C_1 = 4.\) The condition \(y'(0) = -5\) gives, by substituting into the equation for \(y',\)

\[-5C_1 + C_2 - 5C_2 \cdot 0 = -5,\]

or \(-5C_1 + C_2 = -5.\) Since \(C_1 = 4,\) we have \(-20 + C_2 = -5,\) or \(C_2 = 15.\) Therefore

\[y = 4e^{-5t} + 15te^{-5t}.\]
5.) (7 points) Suppose \( y = y(t) \) satisfies \( y'' + 3y' + y = 0 \). Find \( \lim_{t \to \infty} y(t) \).

The characteristic equation is \( r^2 + 3r + 1 = 0 \). This equation does not factor nicely, so we use the quadratic formula to find the roots

\[
 r = \frac{-3 \pm \sqrt{3^2 - 4(1)(1)}}{2} = \frac{-3 \pm \sqrt{5}}{2}.
\]

Hence \( y_1 = e^{\left(-\frac{-3 + \sqrt{5}}{2}\right)t} \) and \( y_2 = e^{\left(-\frac{-3 - \sqrt{5}}{2}\right)t} \) are solutions, and the general solution is

\[
 y = C_1 e^{\left(-\frac{-3 + \sqrt{5}}{2}\right)t} + C_2 e^{\left(-\frac{-3 - \sqrt{5}}{2}\right)t},
\]

for constants \( C_1 \) and \( C_2 \). Note that \( \lim_{t \to \infty} e^{\left(-\frac{-3 + \sqrt{5}}{2}\right)t} = 0 \) because the exponent \( \frac{-3 - \sqrt{5}}{2} \) is negative. Also, since \( \sqrt{5} < 3 \), the exponent \( \frac{-3 + \sqrt{5}}{2} \) is also negative, so \( \lim_{t \to \infty} e^{\left(-\frac{-3 + \sqrt{5}}{2}\right)t} = 0 \). Hence

\[
 \lim_{t \to \infty} y = \lim_{t \to \infty} \left(C_1 e^{\left(-\frac{-3 + \sqrt{5}}{2}\right)t} + C_2 e^{\left(-\frac{-3 - \sqrt{5}}{2}\right)t}\right) = 0.
\]

6.) (13 points) Solve \( y''' + y'' - 5y' + 3y = 0 \).

The characteristic equation is

\[
 r^3 + r^2 - 5r + 3 = 0.
\]

To find a factor, we look for a root of \( p(r) = r^3 + r^2 - 5r + 3 \). Guessing shows that 1 is a root: \( p(1) = 1^3 + 1^2 - 5 \cdot 1 + 3 = 0 \). Hence the root/factor theorem guarantees that \( r - 1 \) divides evenly into \( r^3 + r^2 - 5r + 3 \). Carrying out the division of \( r^3 + r^2 - 5r + 3 \) by \( r - 1 \) gives

\[
 r^3 + r^2 - 5r + 3 = (r - 1)(r^2 + 2r - 3) = (r - 1)(r + 3)(r - 1) = (r - 1)^2(r + 3).
\]

Hence \( r = -3 \) is a root, and \( r = 1 \) is a repeated root. Therefore \( y_1 = e^{-3t} \), \( y_2 = e^t \), and \( y_3 = te^t \) are solutions of the differential equation. By the principle of superposition, so is

\[
 y = C_1 e^{-3t} + C_2 e^t + C_3 te^t,
\]

for constants \( C_1, C_2, \) and \( C_3 \), and this is the general solution.