#1.) /20

1.) (20 points) Solve $y'' - 6y' + 13y = 60 \cos t$.

To find $y_h$, the solution of $y'' - 6y' + 13y = 0$, we look at the characteristic equation $r^2 - 6r + 13 = 0$. We cannot factor this, so using the quadratic formula we get

$$r = \frac{6 \pm \sqrt{36 - 4 \cdot 13}}{2} = \frac{6 \pm \sqrt{-16}}{2} = \frac{6 \pm 4i}{2} = 3 \pm 2i.$$ 

Hence $y_h = C_1 e^{3t} \cos 2t + C_2 e^{3t} \sin 2t$.

To find $y_p$, based on the right side $60 \cos t$ of the equation, we guess $y_p = A \cos t + B \sin t$. Neither term in $y_p$ is a constant multiple of any term of $y_h$, so this guess is satisfactory. Then

$$y_p' = -A \sin t + B \cos t$$

and

$$y_p'' = -A \cos t - B \sin t.$$ 

Substituting these, we get

$$y_p'' - 6y_p' + 13y_p = -A \cos t - B \sin t + 6A \sin t - 6B \cos t + 13A \cos t + 13B \sin t$$

$$= (-A - 6B + 13A) \cos t + (-B + 6A + 13B) \sin t = (12A - 6B) \cos t + (6A + 12B) \sin t.$$ 

Setting this equal to $60 \cos t$ and equating corresponding coefficients, we get the equations $12A - 6B = 60$ and $6A + 12B = 0$. Dividing each equation through by 6 we get $2A - B = 10$ and $A + 2B = 0$. Multiplying the first equation by 2 to get $4A - 2B = 20$ and adding this to the equation $A + 2B = 0$ gives $5A = 20$ or $A = 4$. Then from $A + 2B = 0$ we get $B = -A/2 = -2$. Hence $y_p = 4 \cos t - 2 \sin t$.

Therefore the general solution is $y = y_h + y_p$ or

$$y = C_1 e^{3t} \cos 2t + C_2 e^{3t} \sin 2t + 4 \cos t - 2 \sin t.$$
2.) a.) (5 points) Check that the functions $y_1 = t^2$ and $y_2 = t^{-1}$ are solutions of

$$\tag{*} y'' - \frac{2}{t^2} y = 0, \quad \text{for } t > 0.$$

For $y_1 = t^2$ we have $y_1' = 2t$ and $y_1'' = 2$. Therefore

$$y_1'' - \frac{2}{t^2} y_1 = 2 - \frac{2}{t^2} t^2 = 2 - 2 = 0.$$

Therefore $y_1 = t^2$ is a solution.

For $y_2 = t^{-1}$, we have $y_2' = -t^{-2}$ and $y_2'' = 2t^{-3}$. Therefore

$$y_2'' - \frac{2}{t^2} y_2 = 2t^{-3} - \frac{2}{t^2} t^{-1} = \frac{2}{t^3} - \frac{2}{t^2} \cdot \frac{1}{t} = \frac{2}{t^3} - \frac{2}{t^3} = 0.$$

Therefore $y_2 = t^{-1}$ is a solution.

b.) (15 points) Find the general solution of

$$y'' - \frac{2}{t^2} y = 3t^5, \quad \text{for } t > 0.$$

Warning: The method of undetermined coefficients does not apply to variable coefficient equations.

To find the general solution, we need to find a particular solution $y_p$. Using the method of variation of parameters, we guess a solution of the form $y_p = v_1 y_1 + v_2 y_2$. The equations determining $v_1'$ and $v_2'$ are $y_1 v_1' + y_2 v_2' = 0$ and $y_1' v_1' + y_2' v_2' = g(t)$, where $g(t)$ is the right side of the equation. Using our values of $y_1$, $y_2$ and $g$, we get

$$t^2 v_1' + \frac{1}{t} v_2' = 0$$

$$2 t v_1' - \frac{1}{t^2} v_2' = 3t^5.$$

We multiply the first equation by $-2t$ to get $-2t^3 v_1' - 2v_2' = 0$ and multiply the second equation by $t^2$ to get $2t^3 v_1' - v_2' = 3t^5$. Then when we add these equations, the $v_1'$ terms cancel to give

$$-3v_2' = 3t^5,$$

or $v_2' = -t^5$. Substituting $v_2' = -t^5$ into $-2t^3 v_1' - 2v_2' = 0$ gives $-2t^3 v_1' = 2v_2' = -2t^5$ or $v_1' = t^2$. Integrating the equation $v_2' = -t^5$ gives $v_2 = -t^6/6$ while integrating $v_1' = t^2$ gives $v_1 = t^3/3$ (the constants of integration are not needed here). Hence

$$y_p = v_1 y_1 + v_2 y_2 = \frac{t^3}{3} t^2 - \frac{t^6}{6} \frac{1}{t} = \frac{t^5}{3} - \frac{t^5}{6} = t^5 \left( \frac{1}{3} - \frac{1}{6} \right) = \frac{t^5}{6}.$$  

Then the general solution is $y = y_h + y_p$, or

$$y = C_1 t^2 + C_2 t^{-1} + \frac{t^5}{6}.$$
3.) (10 points) Solve the Cauchy-Euler equation $t^2y'' + ty' + 4y = 0$, for $t > 0$.

Because we have a Cauchy-Euler equation, we try $y = t^r$ as a solution. Then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Substituting into the equation gives

$$t^2r(r-1)t^{r-2} + trt^{r-1} + 4t^r = 0,$$

or

$$r(r-1)t^r + rt^r + 4t^r = 0.$$

Cancelling $t^r$ gives $r(r-1) + r + 4 = 0$, or

$$0 = r^2 - r + 4 = r^2 + 4.$$

Hence $r^2 = -4$ or $r = \pm \sqrt{-4} = \pm 2i$. This is the case of complex roots, and we get solutions $y_1 = t^0 \cos(2 \ln t) = \cos(2 \ln t)$ and $y_2 = t^0 \sin(2 \ln t) = \sin(2 \ln t)$. The general solution $y$ is a linear combination of $y_1$ and $y_2$; that is,

$$y = C_1 \cos(2 \ln t) + C_2 \sin(2 \ln t),$$

for arbitrary constants $C_1$ and $C_2$.

4.) (10 points) Solve $y'' - 2y' = 4e^{2t}$.

First we find the solution $y_h$ of the homogeneous equation $y'' - 2y' = 0$. We guess $y_h = e^{rt}$, which leads to the characteristic equation $r^2 - 2r = 0$. We factor this as $r(r - 2) = 0$, so $r_1 = 0$ and $r_2 = 2$ are the roots. So $y_h = C_1 e^{0t} + C_2 e^{2t} = C_1 + C_2 e^{2t}$.

Now we use the method of undetermined coefficients to try to find a particular solution $y_p$ of the nonhomogeneous problem. For the right side $g(t) = 4e^{2t}$, we guess $y_p = Ae^{2t}$. However, the term $Ae^{2t}$ is a constant multiple of the term $C_2 e^{2t}$ in $y_h$, so we modify our guess for $y_p$ by multiplying by $t$. This gives $y_p = At e^{2t}$. This is not a multiple of any term in $y_h$, so this $y_p$ is satisfactory. So

$$y_p = At e^{2t}.$$

Then

$$y_p' = Ae^{2t} + 2At e^{2t}$$

and

$$y_p'' = 2Ae^{2t} + 2Ae^{2t} + 4At e^{2t} = 4Ae^{2t} + 4At e^{2t}.$$

Substituting this in the left side of the inhomogeneous equation, we get

$$y_p'' - 2y_p' = 4Ae^{2t} + 4At e^{2t} - 2(Ae^{2t} + 2At e^{2t})$$

$$= 4Ae^{2t} + 4At e^{2t} - 2Ae^{2t} - 4At e^{2t} = 2Ae^{2t} = 2Ae^{2t},$$

because the terms involving $te^{2t}$ cancel. Setting this equal to $4e^{2t}$, we get $A = 2$. Therefore $y_p = 2te^{2t}$ and hence the general form of the solution to the equation $y'' - 2y' = 4e^{2t}$ is $y = y_h + y_p$ or

$$y = 2te^{2t} + C_1 + C_2 e^{2t}.$$
5.) (20 points) A spring has mass 2 kg, damping constant 12 N\text{-sec}/m, and spring constant (stiffness) 16 N/m. If it is stretched 1 m beyond its natural length and given an initial velocity of 2 m/sec in the same direction it is stretched, find the time in seconds when the spring reaches its maximum displacement from equilibrium. There is no external force acting on the spring (after the initial velocity is given).

The general equation for a spring with no external force is \( my'' + by' + ky = 0 \). Here \( m = 2 \), \( b = 12 \), and \( k = 16 \). Hence we have \( 2y'' + 12y' + 16y = 0 \). After dividing through by 2, we get

\[
y'' + 6y' + 8y = 0.
\]

The characteristic equation is \( r^2 + 6r + 8 = 0 \). This equation factors as \((r + 2)(r + 4) = 0\). Hence the roots are \( r = -2 \) and \( r = -4 \). Therefore the general equation of the motion of the spring is

\[
y = C_1 e^{-2t} + C_2 e^{-4t},
\]

for arbitrary constants \( C_1 \) and \( C_2 \). We are given that \( y(0) = 1 \) and \( y'(0) = 2 \). From \( y(0) = 1 \) we get \( 1 = C_1 e^0 + C_2 e^0 = C_1 + C_2 \), or

\[
C_1 + C_2 = 1.
\]

We calculate \( y' = -2C_1 e^{-2t} - 4C_2 e^{-4t} \). From \( y'(0) = 2 \) we get \( 2 = -2C_1 e^0 - 4C_2 e^0 = -2C_1 - 4C_2 \), or

\[
-2C_1 - 4C_2 = 2.
\]

Multiplying the equation \( C_1 + C_2 = 1 \) by 2 gives \( 2C_1 + 2C_2 = 2 \). Adding \( 2C_1 + 2C_2 = 2 \) to \(-2C_1 - 4C_2 = 2 \) gives \(-2C_2 = 4 \), so \( C_2 = -2 \). Then from \( C_1 + C_2 = 1 \) we get \( C_1 = 1 - C_2 = 1 - (-2) = 3 \). Therefore

\[
y = 3e^{-2t} - 2e^{-4t}.
\]

The maximum displacement occurs when \( y' = 0 \). Here \( y' = -6e^{-2t} + 8e^{-4t} \). Hence we solve

\[
-6e^{-2t} + 8e^{-4t} = 0,
\]

or \( 8e^{-4t} = 6e^{-2t} \). Multiplying through by \( e^{4t} \) gives \( 8 = 6e^{2t} \), or \( e^{2t} = 8/6 = 4/3 \). Taking the natural logarithm of both sides gives

\[
2t = \ln(e^{2t}) = \ln(4/3),
\]

so

\[
t = \frac{1}{2} \ln(4/3).
\]
6. (20 points) Assuming the fact that the function \( y_1 = e^t \) is a solution of the equation

\[
(*) \quad y'' - \left(2 + \frac{1}{t}\right) y' + \left(1 + \frac{1}{t}\right) y = 0,
\]

find the general solution of \((*)\).

To get the general solution, we need two linearly independent solutions. Since we have one solution, we use the method of reduction of order. We guess a second solution of the form \( y_2 = vy_1 \), where \( v \) is a function of \( t \). Here \( y_2 = \text{ve}^t = \text{ev}^t \); hence by the product rule, \( y'_2 = e^t v + e^t v' \) and \( y''_2 = e^t v + e^t v' + e^t v'' + e^t v'' = e^t v + 2e^t v' + e^t v'' \). Substituting this in the equation, we get

\[
y''_2 - \left(2 + \frac{1}{t}\right) y'_2 + \left(1 + \frac{1}{t}\right) y_2 = 0,
\]

which we can solve by the method of the integrating factor. The integrating factor is \( e^t \)

Setting this equal to 0 and we get the equation \( e^t v'' - \frac{1}{t} e^t v' = 0 \). Dividing through by \( e^t \) we get

\[
v'' - \frac{1}{t} v' = 0.
\]

We make the substitution \( u = v' \). This gives the equation \( u' - \frac{1}{t} u = 0 \), which is a first order linear equation which we can solve by the method of the integrating factor. The integrating factor is

\[
e^{\int -\frac{1}{t} dt} = e^{-\ln t} = e^{\ln(t^{-1})} = t^{-1} = \frac{1}{t}.
\]

Multiplying through the equation \( u' - \frac{1}{t} u = 0 \) by the integrating factor \( \frac{1}{t} \) gives

\[
\frac{1}{t} u' - \frac{1}{t^2} u = 0.
\]

The left side is \( \left(\frac{1}{t} u\right)' \), by the product rule. So we have \( \left(\frac{1}{t} u\right)' = 0 \), which implies \( \frac{1}{t} u = C_1 \), or \( u = C_1 t \). Then since \( u = v' \), we get \( v = \int u = \int C_1 t \, dt = C_1 t^2/2 + C_2 \). Then

\[
y_2 = ve^t = \left(C_1 t^2/2 + C_2\right) e^t = \frac{C_1}{2} t^2 e^t + C_2 e^t.
\]

Since we are just looking for one \( y_2 \), we can take \( C_1 = 2 \) and \( C_2 = 0 \) to obtain \( y_2 = t^2 e^t \). Then the general solution is \( y = C_1 y_1 + C_2 y_2 \), or

\[
y = C_1 e^t + C_2 t^2 e^t.
\]