Please read: I will try to post here a few solutions (or answers). The new solutions will be added to this same file. They might come with no explanation, just the “answer”. If yours do not match mine, you can try to figure out again. (Also, read the disclaimer below!) You can come to office hours if you want explanations for the unexplained answers. Be careful that just because our “answers” were the same, it doesn’t mean that you solved the problem correctly (it might have been a “fortunate” coincidence), and in the exams what matters is the solution itself. I will do my best to post somewhat detailed solutions for harder problems, though.

Disclaimer: I will have to put these solutions together rather quickly, so they are subject to typos and conceptual mistakes. (I expect you to be a lot more careful when doing your HW than I when preparing these.) You can contact me if you think that there is something wrong and I will fix the file if you are correct.

Homework 1

Chapter 1

1. (a) A midpoint of $AB$ is a point $M$ such that $AM \cong MB$. [Note that, a priori, we don’t know if it exists or is unique!]

(b) A perpendicular bisector of a segment $AB$ is a line $l$ which is perpendicular to $\overrightarrow{AB}$ and passing through a midpoint of $AB$.

(c) Given $D$ a point between $A$ and $C$, we have that $\overrightarrow{BD}$ bisects the angle $\angle ABC$, if $\angle ABD \cong \angle DBC$.

(d) Points $A$, $B$ and $C$ are colinear if there exists a line $l$ such that all three lie on $l$. 
(e) Lines \( l, m \) and \( n \) are concurrent if they are distinct and there is a point \( P \) which lies in all three.

2. See the above for similar definitions. E-mail me if you think you want to see them written explicitly.

5. By definition, we have that \( OP \cong OA \) and \( OQ \cong OA \). Thus, as observed in pg. 17, the common notion that “things congruent to the same thing are congruent to each other” tells us that \( OP \cong OQ \).

6. (a) If you think carefully about it, you will see that there is nothing that says anything at all about the notion of between, so there is nothing that can be proved about it. [To actually prove that the statement cannot be proved, we need models, as introduced in Chapter 2.]

(b) This is similar to the above. It cannot be proved.

14. In this problem we can use all known facts of Euclidean geometry without proof, as its intent is to review constructions, not to prove them.

(a) Draw a circle of center \( A \) and radius \( AB \), then a circle of center \( B \) and radius \( BA = AB \). These circles intersect in two points, say \( C \) and \( D \). By construction, \( AC \cong BC \cong AD \cong BD \) [as all are congruent to \( AB \)]. Hence, \( ACBD \) is a rhombus. Thus the line \( \overrightarrow{CD} \) is the desired bisector.

(b) Draw a circle of any radius and center on \( P \). The circle intersects the line in two points having \( P \) as midpoint. Then use (a).

(c) Construct a circle with center \( P \) and radius greater than the distance between \( P \) and \( l \). This will intersect \( l \) in two [distinct] points, say \( A \) and \( B \). By construction \( \triangle PAB \) is isosceles with \( PA \cong PB \) [as both are congruent to the radius]. Then, use (a). [Since the triangle is isosceles, the perpendicular bisector of the base passes through the opposite vertex.]

(d) Using (c), construct a line, say \( m \), perpendicular to \( l \) through \( P \). Now, using (b), construct a line perpendicular to \( m \) passing through \( P \). This line is parallel to \( l \).

(e) Construct a circle with center at the vertex, say \( A \), with any radius. This will intersect the sides of the angle in two distinct points, say \( B \) and \( C \). [Thus, \( \triangle ABC \) is isosceles with base \( BC \).] Now, join \( B \) and \( C \) and use (a) to construct
the perpendicular bisector of $BC$. The part of this line in the interior of the angle is the bisector.

(f) Draw a circle of center $D$ and radius $AC$ and a circle of center $E$ and radius $BC$. These intersect in two points, one in each side of $\overrightarrow{DE}$. Let $F$ be the intersection on the chosen side. [Since $F$ is in the circle of radius $AC$ and center $D$, we have that $DF \cong AC$. Since $F$ is in the circle of radius $BC$ and center $E$, we have that $EF \cong BC$. By SSS, we have that the triangles are congruent.]

(g) Draw a circle with center $B$ and any radius. This intersects the angle $\angle ABC$ in two points, one in ray $\overrightarrow{BA}$, say $P$, and one in $\overrightarrow{BC}$, say $Q$. [Thus $BP \cong BQ$.]

Now, draw a circle with center in $D$ and radius $BQ$. This will intersect $\overrightarrow{DE}$ at a point, say $X$. [Thus, $DX \cong BQ$.] Now, draw a circle of center in $X$ and radius $QP$. The two circles intersect in two points, one in each side of $\overrightarrow{DE}$. Let $F$ be the point in the chosen side. [Then, since $F$ is in the first circle, we have $DF \cong BQ \cong BP$. Since $F$ is in the second circle, we have $XF \cong QP$. By SSS, we have $\triangle BPQ \cong \triangle DXF$, and in particular, $\angle PBQ \cong \angle XDF$. But, by construction, we have $\angle PBQ = \angle ABC$ and $\angle XDF = \angle EDF$.]

Homework 2

Chapter 2

Notes on notation: We shall read “$P \in \mathcal{P}$” as “$P$ is a point” and “$l \in \mathcal{L}$” as “$l$ is a line”. Once can think of $\mathcal{P}$ and $\mathcal{L}$ as the “set of points” and the “set of lines”, respectively, but this is not precise, as in principle we don’t know if all points or lines form a proper set.

We shall also write “$P \in l'$” for “$P$ lies on $l'$”. This, again, is a bit misleading, as for us a line is not [necessarily] a set. [It is an undefined term, after all.]

These are meant to be shortcuts, but one should not take them to mean any more than that!

3. $\forall l \in \mathcal{L}$ s.t. $(\forall P \in \mathcal{P}, ((P \notin l) \land ((\forall m \in \mathcal{L}, (P \notin m) \lor (\exists m \in \mathcal{L} \land (\exists n \in \mathcal{L} \land (m \neq n) \land (P \in m) \land (P \in n) \land (m \parallel l) \land (n \parallel l))))))$

5. (1) By I-3, there are three points not on a line. Say $A$, $B$, $C$.

   (2) By I-1, we have [not necessarily distinct] lines $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$. [We will show that they are distinct.]
(3) Assume $\overrightarrow{AB} = \overrightarrow{BC}$. [RAA hypothesis.]

(4) Then, since, by construction $A, B \in \overrightarrow{AB}$ and $B, C \in \overrightarrow{BC}$, we have, by the previous step, that $A, B, C \in \overrightarrow{AB} = \overrightarrow{BC}$.

(5) This contradicts the first step, and therefore the assumption in step (3) must be false.

(6) Repeat steps (3), (4) and (5), with corresponding adjustments, for the other two pairs of lines. [I’m being lazy here...]

6. [2.3 only] Suppose the statement is false [to use RAA]. This means that there is a line such that all points lie on it. [Practice your negation skills!] But this contradicts I-2. By RAA, the original statement must be true.

9. (a) I-1 is not satisfied, as if you take skew lines, these do not have a plane containing them. I-2,3 are satisfied.

In this case, the model has Euclidean parallel property.

(b) In this case all incidence axioms are satisfied, and has the elliptic parallel property, as two planes through the origin intersect at a line.

(c) This is the Klein circle model discussed in class. It is a model of incidence with the hyperbolic property.

(d) I-1 holds, as given two distinct pairs of antipodal points, there is a unique plane containing these four points and the center of the sphere, giving a unique great circle.

I-2 and I-3 also hold [and it are easier to see].

In this model we have the elliptic parallel property, as two great circles always intersect on a pair of antipodal points.

10. (b) No. Consider two models. First, the tetrahedron one from of Example 3 [on pg. 74]. Second add a point to the 3-point model [from pg. 73], but add an extra point, say $D$ on segment $BC$ and a line $\overrightarrow{AD}$ [which can be associates to the segment $AD$]. One can [rather easily and tediously] check that this is indeed a model for incidence geometry.

The two models are not isomorphic as they have different number of lines, namely the first model has 6, while the latter has only 4. [If they were isomorphic, there would be a bijection between the set of lines, but these implies that the sets must have the same number of elements.]
14. (a) In the 3-point model [of pg. 73], this statement is false, as any pair of lines contains all 3 points. In the usual model of Euclidean geometry [or in $\mathbb{R}^2$], the statement is true. Hence, since we have models of incidence geometry where the statement have different true/false value, neither the statement nor its negation is a theorem in incidence geometry.

17. (a) Let $P$ be the set of beans who are poetical, $F$ be the set of frogs and $D$ be the set of ducks.

- The second statement is $D \cap P$ has some elements.
- The first is $(p \in P \Rightarrow p \notin P)$.

Thus, for the $d \in D \cap P$, of the second statement, we have that in particular, $d \in P$, and from the first statement $d \notin F$ [as $d \in P$]. So, indeed, some ducks are not frogs [namely, those in $D \cap P$].

(b) Let $S$ be the set of things that will silence him, $G$ the set of things that are gold, $H$ the set of things that are heavy, and $L$ the set of things that are light [i.e., not heavy]. Thus, $\sim (x \in H)$ [or $x \notin H$] is logically equivalent to $x \in L$.

- The second statement is $(x \in S \Rightarrow x \in G)$.
- The first is $(x \in G \Rightarrow x \in H)$

By the comment above, the first statement is equivalent to $(x \in G \Rightarrow x \notin L)$. Its contrapositive is $(x \in L \Rightarrow x \notin G)$. The contrapositive of the second is $(x \notin G \Rightarrow x \notin S)$. Thus, $(x \in L \Rightarrow x \notin S)$. Thus, indeed nothing light will silence him.

(c) This statement is false. Assume only lions drink coffee. [This means that although some lions might not drink coffee, if one drinks coffee, then this one is a lion.] Then, no creature that drinks coffee is not fierce.

(d) The statement is again false. Imagine that some, but not all, pillows soft. Then, all non-soft pillows might be pokers, and hence every poker is a [non-soft] pillow.

**Homework 3**

**Chapter 3**

3. (a) Let $P \in AB$. Then, by definition, either $P = A$, $P = B$ or $A * P * B$. 

5
If $P = A$, then $P \in AC$ by definition of $AC$.

If $P = B$, since $A \ast B \ast C$ by hypothesis, we have that $P = B \in AC$ [by definition of $AC$ again].

If $A \ast P \ast B$, since $A \ast B \ast C$ by hypothesis, Proposition 3.3 gives us that $A \ast P \ast C$. Thus, $P \in AC$ by definition of $AC$.

Therefore, if $P \in AB$, then $P \in AC$, and hence $AB \subseteq AC$.

For the second part, remember that by B-1 we have that $A \ast B \ast C$ implies that $C \ast B \ast A$, which allows us to switch the points [or letters] $A$ and $C$ in our proof, obtaining $CB \subseteq CA$.

(b) Let $P \in AC$. [We need to show that $P \in AB$ or $P \in BC$.] If $P = A$ or $P = B$, then $P \in AB \subseteq AB \cup BC$. If $P = C$, then $P \in BC \subseteq AB \cup BC$.

So, suppose $A$, $B$, $C$ and $P$ [still with $P \in AC$] are four distinct points. By I-1 and B-1, we have that all four points are on $l = \overrightarrow{AC}$. [Prove it!]

Let now $D$ be a point not on $\overrightarrow{AC}$, which exist by Proposition 2.3. Then, $P \neq D$, as $P \in \overrightarrow{AC}$, while $D \notin \overrightarrow{AC}$. By I-1, there is a line $m = \overrightarrow{DP}$.

Moreover, $P$ is the only point in both $l$ and $m$: indeed, we’ve seen $P \in l$ above, and $P \in m = \overrightarrow{DP}$ by construction. If there were another point in the intersection, the lines would coincide by I-1. But we have $D \in m = \overrightarrow{DP}$ and $D \notin l$, by the choice of $D$. So, $m \neq l$, and hence there is no other point of intersection [by RAA].

Since $A, B, C \in l$, and distinct from $P$ [as we are assuming now], we have that $A, B, C \notin m$ [as the only point in both is $P$, as seen above]. Also, since $A \ast P \ast C$, we have that $A$ and $C$ are on opposite sides of $m$ [as $AC \cap m = \{P\} \neq \varnothing$]. Since $B \notin l$, we have that either $B$ is either in the same side of $m$ as $A$ or on the same side of $m$ as $C$ [by Proposition 3.2].

If $A$ and $B$ are on the same side of $m$, since $A$ and $C$ are on opposite sides, by the Corollary of B-4, we have that $B$ and $C$ are on opposite sides of $m$. Then, $BC \cap m \neq \varnothing$. But since $BC = l$ and $l \cap m = \{P\}$, we have that $BC \cap m = \{P\}$, and thus $P \in BC$.

If $B$ and $C$ are on the same side of $m$, since $A$ and $C$ are on opposite sides, by the Corollary of B-4, we have that $A$ and $B$ are on opposite sides of $m$. Then,
AB \cap m \neq \emptyset. But since \( \overrightarrow{AB} = l \) and \( l \cap m = \{P\} \), we have that \( AB \cap m = \{P\} \), and thus \( P \in AB \).

(c) From (b), we get that \( AC \subseteq AB \cup BC \). So, we need to show \( AB \cup BC \subseteq AC \).

But in (a) we’ve shown that \( AB \subseteq AC \) and \( BC = CB \subseteq CA = AC \). Thus, \( AB \cup BC \subseteq AC \). Hence, we now only need to show the second part of the statement, i.e., \( AB \cap BC = \{B\} \). Clearly, we have \( B \in AB \cap BC \) [by definition of the segments].

So, suppose that there is \( P \neq B \) such that \( P \in AC \cap BC \). Note that \( P \neq A \), as \( A \not\in BC \), and \( P \neq C \), as \( C \not\in AB \), since \( A \ast B \ast C \), using B-3, and definition of segments. [Fill in the details!]

Then, \( A, B, C \) and \( P \) are four [distinct] colinear points. Consider, as above, a line \( l = \overrightarrow{PD} \), with \( D \not\in \overrightarrow{AB} \). Since \( P \in AB \), with \( P \neq A, B \), we have that \( A \ast P \ast B \), and hence \( A \) and \( B \) are on opposite sides of \( l \). In the same way, since \( P \in BC \), with \( P \neq B, C \), we must have that \( B \) and \( C \) are on opposite sides of \( l \). Thus, by B-4, we have that \( A \) and \( C \) are on the same side of \( l \).

But, since we have \( A \ast P \ast B \) [as seen above] and \( A \ast B \ast C \) [by hypothesis], by Proposition 3.3, we get \( A \ast P \ast C \). This implies [by definition], that \( A \) and \( C \) are on opposite sides of \( l \), contradicting the conclusion of the paragraph above.

Therefore, we must have that \( P = B \) [as \( P \neq B \) led to a contradiction].

5. Let \( P \in \overrightarrow{AB} \). [We must show \( P \in \overrightarrow{AC} \).] If \( P \in AB \), then by Proposition 3.5, we have that \( P \in AC \subseteq \overrightarrow{AC} \). So, suppose that \( P \in \overrightarrow{AB} \setminus AB \). Then, by definition, we have \( A \ast B \ast P \).

We have that \( A, B, C \) and \( P \) are colinear [by I-1 and B-1]. If \( P = C \), then clearly \( P \in AC \subseteq \overrightarrow{AC} \).

So, assume that \( P \neq C \). Since \( A \ast B \ast P \), we also have that \( P \neq A \), and thus by B-3 we have either \( A \ast P \ast C \), \( A \ast C \ast P \), or \( P \ast A \ast C \).

If the first possibility holds, then \( P \in AC \subseteq \overrightarrow{AC} \) [by definition of ray].

If the second possibility holds, then \( P \in \overrightarrow{AC} \) [again by definition].

If the third possibility \([P \ast A \ast C]\) would hold, then since we also have \( P \ast B \ast A \) [as seen in the first paragraph, using B-1], by Proposition 3.3 we would get that \( B \ast A \ast C \), which would be a contradiction to the hypothesis \( A \ast B \ast C \). Thus it cannot occur.
Therefore, we always have that $P \in \overrightarrow{AC}$, and thus $\overrightarrow{AB} \subseteq \overrightarrow{AC}$.

Now, let $P \in \overrightarrow{AC}$. [We must show $P \in \overrightarrow{AB}$.] If $P \in AC$, then by Proposition 3.5, either $P \in AB$ or $P \in BC$. If the former holds, then $P \in \overrightarrow{AAB}$ [by definition of rays].

Suppose then that $P \in BC$, with $P \neq B$ [as $B \in AB$]. If $P = C$, then since $A * B * C$, we have that $P \in \overrightarrow{AB}$ [by definition of rays]. If $P \neq C$ [and since we are assuming that $P \in BC$], we must have $B * P * C$. Since also $A * B * C$ [by hypothesis], we have by Proposition 3.3 and B-1, that $A * B * P$, and $P \in \overrightarrow{AB}$ [by definition of ray].

So, assume now that $P \in \overrightarrow{AC} \setminus AC$. Then, by definition, $A * C * P$. Since we also have $A * B * C$, by definition, we get by Proposition 3.3 that $A * B * P$. Therefore, we get $P \in \overrightarrow{AB}$ [by definition].

So, we always get $P \in \overrightarrow{AB}$, and therefore $\overrightarrow{AC} \subseteq \overrightarrow{AB}$. Since we proved the other inclusion, we get $\overrightarrow{AB} = \overrightarrow{AC}$.

13. Suppose that line $l$ is in the interior of the triangle $\triangle ABC$. By I-2, there are points $P, Q \in l$. Since $l$ is in the interior of the triangle, $P$ and $Q$ must be in the interior of the triangle.

Now, consider the ray $\overrightarrow{PQ}$. Since $P$ is in the interior of the triangle, by Proposition 3.9, we get that the ray intersects one of the sides. Since this point is on the line, but not in the interior [as is on the side], we get a contradiction.

Therefore, a line cannot be contained in the interior of the triangle.

**Note:** One could go further and show that there is a point in $l$ that is in the exterior of the triangle by using B-2.

20. (1) RAA hypothesis.

(2) B-1

(3) Steps 1 and 2.

(4) C-3.

(5) By hypothesis.

(6) Step 4 and C-2.

(7) Step 2 and uniqueness of B-1 [and $F \in \overrightarrow{EF}$].

(8) Steps 3 and 7, and RAA.
22. **Proposition 3.13(b):** Assume that $AB < CD$ and $CD \cong EF$. The former implies, by definition, that there exists $P \in CD \setminus \{C, D\}$ such that $AB \cong CP$. By Proposition 3.12, there exists $Q \in EF \setminus \{E, F\}$ such that $EQ \cong CP$. By C-2 [or, more precisely, by the transitive law for congruences, which was proved using C-2], we have that $AB \cong EQ$. Since $Q \in EF \setminus \{E, F\}$, we have that $AB < EF$ by definition.

**Proposition 3.13(c):** Assume that $AB > CD$ and $CD \cong EF$. The former implies, by definition, that there exists $P \in AB \setminus \{A, B\}$ such that $AP \cong CD$. By C-2 [or, more precisely, by the transitive law for congruences, which was proved using C-2], we have that $AP \cong EF$. Since $P \in AB \setminus \{A, B\}$, we have that $AB > EF$ by definition.

27. We have:

(a) $\angle B \cong \angle C$ [by hypothesis]
(b) $BC \cong CB$ [as $BC = CB$ and reflexive property of congruences, which was proved from C-2]
(c) $\angle C \cong \angle B$ [by hypothesis and reflexive property of congruence of angles, which was proved from C-5]

Thus, by ASA [Property 3.7], we get $\triangle ABC \cong \triangle ACB$. Thus, $AB \cong AC$ [by definition of congruent triangles].

28. Suppose we have $\triangle ABC$ with $\angle A \cong \angle B \cong \angle C$. By Proposition 3.18 [previous problem], since $\angle B \cong \angle C$, we have that $AB \cong AC$.

But also, since $\angle A \cong \angle B$, the same proposition gives that $CA \cong CB$. By the transitivity of congruences, we get that all sides are congruent, and the triangle is equilateral [by definition].

34. (a) Let $AB$ and $CD$ be given segments. By B-2, there exists a point $P$ such that $A \ast B \ast P$. By C-1, there exists a unique $E$ on $\overrightarrow{BP}$ such that $CD \cong BE$.

Since $A \ast B \ast P$, by B-1 and I-1, we have that $\overrightarrow{AB} = \overrightarrow{BP}$. Since $E \in \overrightarrow{BP} \subseteq \overrightarrow{BP} = \overrightarrow{AB}$, we have that $E \in \overrightarrow{AB}$.

Moreover, we claim that $A \ast B \ast E$. Indeed, first observe that $E \neq B$, or there is no segment $BE$ [to be congruent to $CD$]. If $E = P$, then by construction of $P$ we get $A \ast B \ast E$. 

9
So, suppose that \( E \in BP \setminus \{B, P\} \). Then, by definition, we have \( B \ast E \ast P \). Since also \( A \ast B \ast P \) [by choice of \( P \)], Proposition 3.3 [and B-1], we get that \( A \ast B \ast E \). Finally, if \( E \in \overrightarrow{BP} \setminus BP \), then by definition we have \( B \ast P \ast E \). Since, again, we have \( A \ast B \ast P \), the Corollary of Proposition 3.3 gives us that \( A \ast B \ast E \), finishing the proof of the claim.

Hence, the above proves the existence of \( E \) as in the statement of Euclides Axiom 2. We now prove uniqueness.

Let \( E \) be as above and suppose we have \( E' \) such that \( A \ast B \ast E' \) and \( CD \cong BE' \).

[We must show that \( E' = E \).] The betweeness condition above on \( E' \) gives us that \( E' \notin \overrightarrow{BA} \) [prove it!], and by Propositions 3.4 we have that \( E' \) lies on the opposite ray of \( \overrightarrow{BA} \). By construction of \( P \), this opposite ray is \( \overrightarrow{BP} \) [prove it!]. So, \( E' \in \overrightarrow{BP} \). Since also \( BE' \cong CD \) [by assumption], we have that the uniqueness of \( C-1 \) gives us that \( E = E' \).

\[(b) \text{Let } \gamma \text{ denote the a circle which is, simultaneously, a circle of center } O \text{ and radius } OA \text{ and of center } O' \text{ and radius } O'A'. \text{[This means that if } P \in \gamma, \text{ then } OP \cong OA \text{ and } O'P \cong O'A', \text{ by definition of center/radius of a circle.]}\]

Suppose \( O \neq O' \). By part (a), let \( P \) such that \( O' \ast O \ast P \) with \( OP \cong OA \). Thus, \( P \in \gamma \) by as it has center \( O \) and radius \( OA \). Also, let \( P' \) such that \( O \ast O' \ast P' \) such that \( O'P' \cong O'A' \). Then, \( P' \in \gamma \), as it has also center \( O' \) and radius \( \gamma' \).

By B-3, we must have \( P \neq P' \), and by B-1 and I-1, we have that \( P, O, P' \) and \( O' \) are colinear.

Now, since \( P \ast O \ast O' \), by definition [and C-2], we have that \( O'P > OP \). Since also \( O \ast O' \ast P' \), by definition [and C-2], we have that \( OP > O'P' \).

Since \( P, P' \in \gamma \), we have \( OP, OP \cong OA \) [center \( O \) and radius \( OA \)] and \( O'P, O'P' \cong O'A' \) [center \( O' \) and radius \( O'A' \)]. By Proposition 3.13 parts (c) and (d) [used multiple times], we have that the above paragraph yields \( O'P \cong O'A' > OA \cong OP \) and \( OP' \cong OA > O'A' \cong O'P' \). But this contradicts uniqueness of Proposition 3.13(a), and we have a contradiction. Thus, we must have \( O = O' \).

So, assume that \( OA \) and \( OA' \) are both radii of \( \gamma \) [of center \( O \)]. [We need to show \( OA \cong OA' \).] But then, since \( A \in \gamma \) [by definition of a circle of center \( O \) and radius \( OA \)]. Then, by definition of a circle of center \( O \) and radius \( OA' \), we must have \( OA \cong OA' \).