Please read: I will try to post here a few solutions (or answers). The new solutions will be added to this same file. They might come with no explanation, just the “answer”. If yours do not match mine, you can try to figure out again. (Also, read the disclaimer below!) You can come to office hours if you want explanations for the unexplained answers. Be careful that just because our “answers” were the same, it doesn’t mean that you solved the problem correctly (it might have been a “fortunate” coincidence), and in the exams what matters is the solution itself. I will do my best to post somewhat detailed solutions for harder problems, though.

Disclaimer: I will have to put these solutions together rather quickly, so they are subject to typos and conceptual mistakes. (I expect you to be a lot more careful when doing your HW than I when preparing these.) You can contact me if you think that there is something wrong and I will fix the file if you are correct.

Homework 4

Section 4.1

1. (b) Because the results of the sum and scalar multiplication are ordered pairs.
   (c) Using the list from pg. 172: 1 to 5, as they do not involve scalar multiplication.
   (d) For 7:

   \[ k(u + v) = k((u_1, u_2) + (v_1, v_2)) = k(u_1 + v_1, u_2 + v_2) \]
   \[ = (0, k(0, u_2 + v_2)) = (0, ku_2 + kv_2) = (0, ku_2) + (0, kv_2) = ku + kv. \]
For 8:

\[(k + m)u = (k + m)(u_1, u_2) = (0, (k + m)u_2) = (0, ku_2 + mu_2) = ku + mu.\]

For 9:

\[k(mu) = k(m(u_1, u_2)) = k(0, mu_2) = (0, kmu_2) = (km)u\]

(d) Take \(u = (1, 1)\). Then \(1u = (0, 1 \cdot 1) = (0, 1) \neq (1, 1) = u\).

4. It is a vector space. With the what we’ve learned from Section 4.2, since \(\mathbb{R}^2\) is a vector space and we are using the same operations, we only need to check three things:

- 0 is in the set: It is true in this case as \(0 = (0,0)\) and so it is in our set [as the second coordinate is zero].
- Closed under addition: Taking two elements in the set, say \((x_1, 0)\) and \((x_2, 0)\), we have \((x_1, 0) + (x_2, 0) = (x_1 + x_2, 0)\), which is also in the set [as the second coordinate is zero].
- Closed under scalar multiplication: Take an element of the set, say \((x, 0)\) and a scalar \(k \in \mathbb{R}\). Then, \(k(x, 0) = (kx, 0)\) is also in the set [as the second coordinate is zero].

So, the set is a vector space. [In this section we actually would have to check axioms 1, 4, 5, 6 from pg. 172, but since you don’t need to do that in the test, I am giving the shorter solution that you should give in the test.]

8. It is not a vector space. For instance \(I_2\) [the identity] is invertible, but \(0 \cdot I_2\) is the zero matrix, which is not invertible. So, axiom 6 [from pg. 172] fails. [So, does axiom 1, as \(I_2\) and \(-I_2\) are both invertible, but they add to the zero matrix, which is not.]

9. It is a vector space. Again, we use the method of Section 4.2:

- 0 is in the set: Just take \(a = b = 0\).
- Closed under addition: We have

\[
\begin{bmatrix}
  a_1 & 0 \\
  0 & b_1
\end{bmatrix}
+ \begin{bmatrix}
  a_2 & 0 \\
  0 & b_2
\end{bmatrix}
= \begin{bmatrix}
  a_1 + a_2 & 0 \\
  0 & b_1 + b_2
\end{bmatrix}
\]

is in the set [as it is diagonal].
• Closed under scalar multiplication: We have

\[
k \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} ka & 0 \\ 0 & kb \end{bmatrix}
\]

is in the set [as it is diagonal].

10. It is a vector space. Again, we use the method of Section 4.2:

• \(0\) is in the set: The function constant equal to zero is clearly in \(V\), since its value at any \(x\) is 0, so in particular its value at \(x = 1\) is 0.

• Closed under addition: If \(f(1) = 0\) and \(g(1) = 0\), then \((f + g)(1) = f(1) + g(1) = 0 + 0 = 0\). So, if \(f, g \in V\), then \(f + g \in V\).

• Closed under scalar multiplication: If \(f(1) = 0\) and \(k \in \mathbb{R}\), then \((kf)(1) = kf(1) = k0 = 0\). So, if \(f \in V\), then \(kf \in V\).

11. It is a vector space. Since the sum and product are not the standard ones [from \(\mathbb{R}^2\)] we need to check all 10 properties from pg. 172. Let’s call the set in question [ordered pairs with first coordinate equal to 1] \(V\). [We will use \(\mathbf{u} = (1, y)\), \(\mathbf{v} = (1, y')\), and \(\mathbf{w} = (1, y'')\) as generic elements.]

(1) This is clear as when we sum, the first coordinate remains equal to one.

(2) \((1, y) + (1, y') = (1, y + y') = (1, y') + (1, y)\).

(3) \((1, y) + ((1, y') + (1, y'')) = (1, y) + (1, y' + y'') = (1, y + (y' + y'')) = (1, (y + y') + y'') = (1, y + y') + (1, y'') = (1 + y) + (1, y')) + (1 + y').

(4) The zero is \((1, 0)\), as \((1, 0) + (1, y) = (1, 0 + y) = (1, y)\).

(5) The negative of \((1, y)\) is \((1, -y)\), as \((1, y) + (1, -y) = (1, y - y) = (1, 0)\).

(6) This is clear as the scalar multiplication leaves the first coordinate equal to one.

(7) \(k((1, y) + (1, y')) = k(1, y + y') = (1, ky + ky') = (1, ky) + (1, ky') = k(1, y) + k(1, y')\).

(8) \((k + m)(1, y) = (1, (k + m)y) = (1, ky + my) = (1, ky) + (1, my) = k(1, y) + m(1, y)\).

(9) \(k(m(1, y)) = k(1, my) = (1, k(my)) = (1, (km)y) = (km)(1, y)\).

(10) \(1(1, y) = (1, 1 \cdot y) = (1, y)\).
True-False:

(a) False. [See (c).]

(b) False. [See (c).]

(c) True.

(d) False. One of the elements of a vector space must be the zero vector $0$. Suppose we have another vector $v \neq 0$. Then, what happens with $2v$? It cannot be equal to $0$ as by Theorem 4.1.1(d), if $2v = 0$, then either $2 = 0$ or $v = 0$, as neither is true. It also cannot be $v$ as if $2v = v$, then adding $-v$ [from axiom 5] from both sides gives

$$2v - v = v - v$$
$$2v - 1v = 0 \quad \text{[by axioms 10 and 5]}$$
$$(2 - 1)v = 0 \quad \text{[by axiom 7]}$$
$$1v = 0$$
$$v = 0 \quad \text{[by axiom 10]}$$

But this is false, so $2v$ cannot be equal to $v$. Therefore, $v$ is a third [distinct] vector, and hence we cannot have only two vectors in a vector space. [We either have only one and $V = \{0\}$, or we actually have infinitely many, as all multiples of a non-zero vector are distinct!]

(e) False. For instance, axiom 1 fails: $x + 1$ and $-x$ both have degree exactly one, but their sum 1, which has degree zero.

Section 4.2

1. (a) Yes. [The criterion to be on the set here is that the second and third coordinates are zero.]
   - $0$ is in the set: Just take $a = 0$.
   - Closed under addition: $(a, 0, 0) + (b, 0, 0) = (a + b, 0, 0)$ is in the set.
   - Closed under scalar multiplication: $k(a, 0, 0) = (ka, 0, 0)$ is in the set.

(b) No. [The criterion to be on the set here is that the second and third coordinates are one.] All properties fail. You need to check only one. I will check them all here just to show you how one would do it.
• 0 is not in the set: No choice of \(a\) gives \((0, 0, 0)\), as the last two coordinates are always ones.

• Not closed under addition: We have that \((1, 1, 1)\) is in the set, but \((1, 1, 1) + (1, 1, 1) = (2, 2, 2)\) is not in the set.

• Not closed under scalar multiplication: We have that \((1, 1, 1)\) is in the set, but \(2 \cdot (1, 1, 1) = (2, 2, 2)\) is not in the set.

(c) Yes. [The criterion to be on the set here is that the sum of the first and third coordinates are equal to the first.]

• 0 is in the set: \((0, 0, 0)\) is in the set as \(0 = 0 + 0\).

• Closed under addition: Suppose that \((a, b, c)\) and \((d, e, f)\) are in the set, i.e., \(b = a + c\) and \(e = d + f\). Then, \((a, b, c) + (d, e, f) = (a + d, b + e, c + f)\), and \(b + e = (a + c) + (d + f) = (a + d) + (c + f)\).

• Closed under scalar multiplication: Suppose that \((a, b, c)\) is in the set, i.e., \(b = a + c\). Then, \(k(a, b, c) = (ka, kb, kc)\), and \(kb = k(a + c) = ka + kc\).

2. (d) Yes.

• 0 is in the set: Clearly the zero matrix is symmetric.

• Closed under addition: Suppose that \(A\) and \(B\) are in the set, i.e., \(A^T = A\) and \(B^T = B\). Then, by properties of the transpose, \((A + B)^T = A^T + B^T = A + B\), i.e., \((A + B)^T = A + B\), and so \(A + B\) is symmetric.

• Closed under scalar multiplication: Suppose that \(A\) is in the set, i.e., \(A^T = A\). Then, by properties of the transpose, \((kA)^T = kA^T = kA\), i.e., \((kA)^T = kA\), and so \(kA\) is symmetric.

(e) Yes.

• 0 is in the set: Clearly the zero matrix is in the set.

• Closed under addition: Suppose that \(A\) and \(B\) are in the set, i.e., \(A^T = -A\) and \(B^T = -B\). Then, by properties of the transpose, \((A + B)^T = A^T + B^T = -A + (-B) = -(A + B)\), i.e., \((A + B)^T = -(A + B)\), and so \(A + B\) is in the set.

• Closed under scalar multiplication: Suppose that \(A\) is in the set, i.e., \(A^T = -A\). Then, by properties of the transpose, \((kA)^T = kA^T = k \cdot (-A) = -(kA)\), i.e., \((kA)^T = -kA\), and so \(kA\) is symmetric.
3. (a) Yes. [A “generic” element in this case is of the form \(a_1 x + a_2 x^2 + a_3 x^3\), and the
criterion is that the term free of \(x\) is zero.]

- \(0\) is in the set: Clearly the zero is in the set. [Take \(a_1 = a_2 = a_3 = 0\].
- Closed under addition: We have that \((a_1 x + a_2 x^2 + a_3 x^3) + (b_1 x + b_2 x^2 + b_3 x^3) =\)
  \((a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3\) is in the set.
- Closed under scalar multiplication: We have that \(k(a_1 x + a_2 x^2 + a_3 x^3) =\)
  \((ka_1)x + (ka_2)x^2 + (ka_3)x^3\) is in the set.

(b) Yes. [The criterion is that the sum of the coefficients is zero.]

- \(0\) is in the set: Clearly the zero is in the set. [Take \(a_0 = a_1 = a_2 = a_3 = 0\]
  and then \(a_0 + a_1 + a_2 + a_3 = 0\].
- Closed under addition: Suppose that \(a_0 + a_1 x + a_2 x^2 + a_3 x^3\) and \(b_0 + b_1 x +\)
  \(b_2 x^2 + b_3 x^3\) are in the set, i.e., \(a_0 + a_1 + a_2 + a_3 = 0\) and \(b_0 + b_1 + b_2 + ab_3 = 0\].
  Then, \((a_0 + a_1 x + a_2 x^2 + a_3 x^3) + (b_0 + b_1 x + b_2 x^2 + b_3 x^3) = (a_0 + b_0) +\)
  \((a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3\). Since \((a_0 + b_0) + (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) =\)
  \((a_0 + a_1 + a_2 + a_3) + (b_0 + b_1 + b_2 + b_3) = 0 + 0 = 0\), the sum is in the set.
- Closed under scalar multiplication: Suppose that \(a_0 + a_1 x + a_2 x^2 + a_3 x^3\) is
  in the set, i.e., \(a_0 + a_1 + a_2 + a_3 = 0\]. Then, \(k(a_0 + a_1 x + a_2 x^2 + a_3 x^3) =\)
  \((ka_0) + (ka_1)x + (ka_2)x^2 + (ka_3)x^3\). Since \((ka_0) + (ka_1) + (ka_2) + (ka_3) =\)
  \(k(a_0 + a_1 + a_2 + a_3) = k \cdot 0\), the scalar multiple is in the set.

True-False:

(a) True.
(b) True.
(c) False. For example, the subset \(\{(0, 0), (1, 0)\}\) of \(\mathbb{R}^2\) has the zero of \(\mathbb{R}^2\), but it is not a
subspace of \(\mathbb{R}^2\). [For instance, \(2 \cdot (1, 0) = (2, 0)\) is not in the set.]
(d) It depends... The actual answer should be false, as in principle \(\mathbb{R}^2\) is not a subset
of \(\mathbb{R}^3\). On the other hand, one can think of \(\mathbb{R}^2\) inside of \(\mathbb{R}^3\) as the \(xy\)-plane, i.e.,
\(\{(x, y, 0) : x, y \in \mathbb{R}\}\). [This is a common idea.] In that case, the answer is true.
(e) False. This is only true if the system is homogeneous [i.e., \(b = 0\)]. If \(b \neq 0\), then
\(x = 0\) is not a solution and hence the set of solutions does not have the zero vector,
and therefore is not a subspace.
(f) True. [It is a subspace!]

(g) True.

- 0 is in the set: Since it is in each subspace, it is in the intersection.
- Closed under addition: Take two elements of the intersection. So, in particular, they are in the first subspace. So, adding these we get an element of the first subspace, as it is a closed under addition. But these two elements, being in the intersection, are also in the second subspace. Then, in a similar way, their sum is also in the second subspace. Therefore, since the sum is both subspaces, it is in the intersection.
- Closed under scalar multiplication: Take an element of the intersection. So, in particular, it is in the first subspace. So, multiplying it by a scalar we get an element of the first subspace, as it is a closed under scalar multiplication. But since this element is in the intersection, it is also in the second subspace. Then, in a similar way, its scalar multiplication by the same scalar is also in the second subspace. Therefore, since it is both subspaces, it is in the intersection.

(h) False. For instance, take two distinct lines through the origin in \( \mathbb{R}^2 \). Then, each line is a subspace of \( \mathbb{R}^2 \), but the union is not a subspace. [Remember: we have seen that all subspaces of \( \mathbb{R}^2 \) are the origin, lines through the origin, and all of \( \mathbb{R}^2 \).]

(i) False. In \( \mathbb{R}^2 \) we have that \( \text{span} \{(1, 0)\} = \text{span} \{(-1, 0)\} \) [they both give the x-axis], but the sets are different.

(j) True. The zero matrix is upper triangular and we have seen the sums and scalar multiples of upper triangular matrices are upper triangular also.

(k) False. For instance, the polynomial 1 [i.e., constant equal to one] is no in that span, for if it were, then

\[
1 = k_1(x - 1) + k_2(x - 1)^2 + k_3(x - 1)^3
\]

for some \( k_1, k_2, k_3 \in \mathbb{R} \). But, if that were the case, plugging \( x = 1 \) would give us:

\[
1 = k_1(1 - 1) + k_2(1 - 1)^2 + k_3(1 - 1)^3 = 0,
\]

which is not true. Hence, there are no such real numbers \( k_1, k_2, k_3 \).