1) [10 points] Put the following matrix in reduced row echelon form:

\[
\begin{bmatrix}
0 & 1 & 1 & 0 & 2 \\
2 & 0 & 4 & 0 & 2 \\
3 & 1 & 7 & 0 & 5 \\
0 & 1 & 1 & 1 & 3
\end{bmatrix}
\]

Solution.

\[
\begin{bmatrix}
0 & 1 & 1 & 0 & 2 \\
2 & 0 & 4 & 0 & 2 \\
3 & 1 & 7 & 0 & 5 \\
0 & 1 & 1 & 1 & 3
\end{bmatrix} \sim
\begin{bmatrix}
1 & 0 & 2 & 0 & 1 \\
0 & 1 & 1 & 0 & 2 \\
3 & 1 & 7 & 0 & 5 \\
0 & 1 & 1 & 1 & 3
\end{bmatrix} \sim
\begin{bmatrix}
1 & 0 & 2 & 0 & 1 \\
0 & 1 & 1 & 0 & 2 \\
0 & 1 & 1 & 0 & 2 \\
0 & 1 & 1 & 1 & 3
\end{bmatrix} \sim
\begin{bmatrix}
1 & 0 & 2 & 0 & 1 \\
0 & 1 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1
\end{bmatrix} \sim
\begin{bmatrix}
1 & 0 & 2 & 0 & 1 \\
0 & 1 & 1 & 0 & 2 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
2) [10 points] Let

$$A = \begin{bmatrix}
1 & 1 & 0 & 2 \\
0 & 0 & 2 & -2 \\
1 & -1 & 3 & 2 \\
1 & 2 & 1 & 2 \\
\end{bmatrix}$$

Compute $\det(A)$.

**Solution.**

$$\begin{vmatrix}
1 & 1 & 0 & 2 \\
0 & 0 & 2 & -2 \\
1 & -1 & 3 & 2 \\
1 & 2 & 1 & 2 \\
\end{vmatrix} = -2 \begin{vmatrix}
1 & 1 & 2 \\
1 & -1 & 2 \\
1 & 2 & 2 \\
\end{vmatrix} - 2 \begin{vmatrix}
1 & 1 & 0 \\
1 & -1 & 3 \\
1 & 2 & 1 \\
\end{vmatrix}$$

$$= -2 \cdot 0 + -2((-1 + 3 + 0) - (0 + 6 + 1)) = 10.$$
3) [40 points] You should be able to answer the following questions quickly. Give short justifications [or show a little work] unless stated otherwise.

(a) [4 points] What is the dimension of $P_5$ [where $P_5$ is the vector space of polynomials of degree at most 5]? [No need to justify.]

Answer. 6 [as $\{1, x, x^2, x^3, x^4, x^5\}$ is a basis].

(b) [4 points] Is $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ -3 & 1 \end{bmatrix}, \begin{bmatrix} -2 & -5 \\ 6 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$ linearly independent in $M_{2\times2}$ [i.e., in the vector space of $2 \times 2$ matrices]? Justify your answer in one short sentence.

Answer. No, as we have more vectors, namely 5, than the dimension of $M_{2\times2}$, namely 4.

(c) [4 points] If $A = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$, then find $A^{-1}$. [No need to show work.]

Answer. $\frac{1}{3} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$.

(d) [4 points] Give the matrix that represents the rotation by $\pi/2$ about the $x$-axis, followed by a projection onto the $xy$-plane in $\mathbb{R}^3$. [No need to justify.]

Answer.

$e_1 \rightarrow e_1 \rightarrow e_1$
$e_2 \rightarrow e_3 \rightarrow 0$
$e_3 \rightarrow -e_2 \rightarrow -e_2$

and so,

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$
(e) [4 points] If $Ax = b$ has no solution, then what can we say about the reduced echelon form of $A$. [No need to justify.]

**Answer.** It has a row of zeros. □

(f) [4 points] Let $S, T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be linear transformations given by

\[
T(x_1, x_2, x_3, x_4) = (x_1, x_3, x_2, x_4),
\]

\[
S(x_1, x_2, x_3, x_4) = (x_1 + x_2, 2x_1 - x_3, 0, x_1 + x_2 + x_4).
\]

Give $[T \circ S]$.

**Answer.** We have

\[
[T] = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad [S] = \begin{bmatrix}
1 & 1 & 0 & 0 \\
2 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1
\end{bmatrix},
\]

and so,

\[
[T \circ S] = [T] \cdot [S] = \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
2 & 0 & -1 & 0 \\
1 & 1 & 0 & 1
\end{bmatrix}
\]

\□

(g) [4 points] Let $v = (1, 0, 0, 1)$ and $w = (2, -1, 3, 1)$. Find the component of $w$ orthogonal to $v$.

**Answer.** We have $\text{proj}_v w = \frac{v \cdot w}{\|v\|^2}v = \frac{3}{2}(1, 0, 0, 1)$, and the orthogonal component is then $(2, -1, 3, 1) - \left(\frac{3}{2}, 0, 0, \frac{3}{2}\right) = (1/2, -1, 3, -1/2)$. □
(h) [4 points] If $A$ is an invertible matrix, with $A^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$ [note that this is the inverse of $A$, not $A!$], find the solutions [if any] of $A\mathbf{x} = \mathbf{b}$ for $\mathbf{b} = (1, 0)$ and $\mathbf{b} = (2, 1)$. [So, there are two systems to solve!]

Answer. Since $A$ is invertible, the solution is $A^{-1}\mathbf{b}$. So, the solutions are $(1, -1)$ and $(4, -3)$. 

(i) [4 points] Let $S = \{(1, 0, 1), (-2, 1, 1), (0, 0, 3)\}$, and $V = \text{span}(S)$. Describe how you would find if $\mathbf{v} = (1, -3, 1)$ is in $V$. More precisely, set up a system [in matrix form!] and say how solving the system would tell you if $\mathbf{v} \in V$ or not. [The answer is short!]

Answer. We have that $\mathbf{v} \in V$ if, and only if, the system

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

has a solution. 

(j) [4 points] If $A$ is a 5 by 4 matrix of nullity 3, give the rank of $A$ and the nullity of $A^T$. [No need to justify.]

Answer. Since the nullity plus the rank is the number of columns, we have that the rank of $A$ is $4 - 3 = 1$. Since the rank of $A$ is the same as the rank of its transpose, we have that the nullity of $A^T$ is $5 - 1 = 4$. 

5
4) [15 points] Let
\[ A = \begin{bmatrix} 3 & 1 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}. \]

(a) [4 points] Show that the eigenvalues of \( A \) are \(-1\) and \(3\).

Solution. The characteristic equation is given by
\[
\det(\lambda I_3 - A) = \begin{vmatrix} (\lambda - 3) & -1 & -3 \\ 0 & (\lambda + 1) & 0 \\ 0 & 0 & (\lambda - 3) \end{vmatrix} = (\lambda - 3)(\lambda + 1)(\lambda - 3) = 0.
\]
Thus, the eigenvalues are the solutions \( \lambda = 3, -1 \).

(b) [4 points] Find the eigenspace associated to the eigenvalue \(3\). [Since I gave you the eigenvalue, you can do this part even if you could not do part (a).]

Solution. For the eigenvalue \( \lambda = 3 \), we need to find the null space of
\[
3I_3 - A = \begin{bmatrix} 0 & -1 & -3 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]
which has reduced echelon form
\[
\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.
\]
So, the eigenspace is \( \text{span}\{(1, 0, 0)\} \).
5) [10 points] Is the set $V$ of all 2 by 2 diagonal matrices [with the usual sum and scalar multiplication of matrices] a subspace of $M_{2 \times 2}$ [i.e., of the vector space of all 2 by 2 matrices]? [Justify!]

Solution. Yes! Since it is a subspace, we only need to check that it is closed under addition and scalar multiplication.

Let $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} \in V$ and $k \in \mathbb{R}$. Then,

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} a + c & 0 \\ 0 & b + d \end{bmatrix}$$

and

$$k \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} ka & 0 \\ 0 & kb \end{bmatrix}.$$  

Since both are in $V$ [i.e., both are diagonal], we have that $V$ is a subspace. \(\square\)
6) [20 points] Let
\[ v_1 = (-3, -3, -7, -34, -11, 3, -33), \]
\[ v_2 = (2, 2, 4, 20, 6, -2, 20), \]
\[ v_3 = (1, 1, 2, 10, 3, -1, 10), \]
\[ v_4 = (2, 2, 5, 24, 8, -1, 21), \]
\[ v_5 = (-1, -1, -4, -18, -7, -1, -12), \]
and \( S = \{v_1, v_2, v_3, v_4, v_5\} \), and \( V = \text{span}(S) \). Given that
\[
\begin{bmatrix}
  v_1 \\
v_2 \\
v_3 \\
v_4 \\
v_5
\end{bmatrix} \sim \begin{bmatrix}
  1 & 1 & 0 & 2 & -1 & 0 & 2 \\
  0 & 1 & 4 & 2 & 0 & 3 \\
  0 & 0 & 0 & 0 & 1 & -2 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
\[
\begin{bmatrix}
  v_1 & v_2 & v_3 & v_4 & v_5
\end{bmatrix} \sim \begin{bmatrix}
  1 & 1 & 0 & 0 & 0 \\
  0 & 1 & 1/2 & 0 & 3/2 \\
  0 & 0 & 0 & 0 & 1 \\
  0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]
answer the questions below. [You do not need to justify any of the items below.]

(a) [4 points] What are the dimension of \( V \) and \( V^\perp \) [the orthogonal complement of \( V \) in \( \mathbb{R}^7 \)]?

\textit{Solution.} They are 3 and 4 respectively. \( \square \)

(b) [4 points] Find a basis for \( V \).

\textit{Solution.} We can take the first three rows of the first matrix in echelon form, say \( \{w_1, w_2, w_3\} \) or \( \{v_1, v_2, v_4\} \). \( \square \)
(c) [4 points] Describe $V^\perp$ as a matrix space [row space, column space, null space] of some matrix. [In other words, fill in the blanks of “$V^\perp$ is the _____ space of the matrix _____.”]

Solution. $V^\perp$ is the null space of the first matrix in echelon form [or of the matrix with the $v_i$’s as rows].

(d) [4 points] Find the coordinates of each of the vectors in $S$ with respect to the basis you’ve found in item (b).

Solution. If we use the basis $B = \{w_1, w_2, w_3\}$, then

$$(v_1)_B = (-3, -7, 3) \quad (v_2)_B = (2, 4, -2)$$
$$(v_3)_B = (1, 2, -1) \quad (v_4)_B = (2, 5, -1)$$
$$(v_5)_B = (-1, -4, -1).$$

If we use $B’ = \{v_1, v_2, v_4\}$, then

$$(v_1)_{B’} = (1, 0, 0) \quad (v_2)_{B’} = (0, 1, 0)$$
$$(v_3)_{B’} = (0, 1/2, 0) \quad (v_4)_{B’} = (0, 0, 1)$$
$$(v_5)_{B’} = (0, 3/2, -2).$$

(e) [4 points] Which vectors from the standard basis of $\mathbb{R}^7$ you can add to the vectors in the basis of $V$ you’ve in (b) to obtain a basis of all of $\mathbb{R}^7$?


\[9\]