Solutions to (Selected) HW Problems

Math 251 – Summer 2009

June 30, 2009

Please read: I will *try* to post here a few solutions (or answers). The new solutions will be added to this same file. They might come with no explanation, just the "answer". If yours do not match mine, you can try to figure out again. (Also, read the disclaimer below!) You can come to office hours if you want explanations for the unexplained answers. Be careful that just because our "answers" were the same, it doesn't mean that you solved the problem correctly (it might have been a "fortunate" coincidence), and in the exams what matters is the solution itself. I will do my best to post somewhat detailed solutions for harder problems, though.

Disclaimer: I will have to put these solutions together rather quickly, so they are subject to typos and conceptual mistakes. (I expect you to be a lot more careful when doing your HW than I when preparing these.) You can contact me if you think that there is something wrong and I will fix the file if you are correct.

Homework 1

Section 1.1

2. All three. [Note that for (b), we must have $k \neq 0$, for the equation to make sense.]

11. If $k = 6$, then the lines $x - y = 3$ and $2x - 2y = k$ coincide, yielding infinitely many solutions, i.e., all points in the line.

On the other hand, if $k \neq 6$, then the lines have the same slope and are distinct: the slope of both lines is 1, and so they are parallel[†]; but since the first line passes through

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[†] Remember that, for us, parallel does not exclude the possibility of the two lines coincide!

$(3,0)$ [since $3-0=3$, but the second doesn't [since $2\cdot 3-2\cdot 0=6\neq k$, by assumption], they do not coincide. Hence, there is no common solution to both equations, which means that the system has no solutions.

Note that there is never a single solution since the lines are always parallel [independently of the value of k].

13. If the system is consistent, the lines intersect. So, if two of those lines are parallel, they must coincide. If this indeed happens, then one can discard one of the equations of the coinciding lines, since they give the same solutions.

If no two lines coincide, then none pair of lines have the same slope, and therefore each pair crosses in a single point. But, since there is a solution, all lines must pass through one common point, which gives us the solution. Discarding then any of the lines would still give us two lines that cross at the same point as before. [Draw a picture.] So, the new system has exactly the same solution.

Section 1.2

5. (a)

$$\begin{aligned} x_1 &= -37,\\ x_2 &= -8,\\ x_3 &= 5. \end{aligned}$$

(c)

$$\begin{aligned} x_1 &= -11-7s+2t,\\ x_2 &= s,\\ x_3 &= -4-3t,\\ x_4 &= 9-3t,\\ x_5 &= t, \end{aligned}$$

for all $s,t\in\mathbb{R}$.

7.(b)

$$\begin{aligned} x_1 &= -1/7 - 3t/7, \\ x_2 &= 1/7 - 4t/7, \\ x_3 &= t, \end{aligned}$$

for all $t \in \mathbb{R}$.

10.(b) Inconsistent. The augmented matrix reduces to the following reduced echelon form:

$$\left[\begin{array}{cccc|c} 1 & 0 & \frac{17}{5} & -\frac{8}{5} & 0 \\ 0 & 1 & \frac{6}{5} & \frac{6}{5} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

17. Gauss elimination gives:

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 10/7 \\ 0 & 0 & (16-a^2) & 4-a \end{array}\right]$$

If $a = -4$, then we get $0 = -8$ in the last row, and there would be no solution.

If $a = 4$, we get $0 = 0$ in the last row, and no leading one in the third column. So, z would be a free variable, giving us infinitely many solutions.

If $a \neq \pm 4$, then $(16 - a^2) \neq 0$, and we would then get:

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 10/7 \\ 0 & 0 & 1 & a+4 \end{array}\right]$$

and the system would have a single solution.

21. Let $x = \sin(\alpha)$, $y = \cos(\beta)$, $z = \tan(\gamma)$. Then, we have a linear system in x, y, and z, associated to the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 5 & 3 & 0 \\ -1 & 5 & 5 & 0 \end{array}\right],$$

which reduces to

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

So, the only solution is $x = y = z = 0$, i.e., $\sin(\alpha) = \cos(\beta) = \tan(\gamma) = 0$. Since $0 \leq \alpha, \beta, \gamma \leq 2\pi$, we have

$$\begin{aligned} \alpha &= 0, \pi, \text{ or } 2\pi, \\ \beta &= \pi/2 \text{ or } 3\pi/2, \\ \gamma &= 0, \pi \text{ or } 2\pi. \end{aligned}$$

Combining all possibilities gives us 18 possible solutions.

[**Note:** The text has a typo, as the statement says "$0 \leq \gamma < 2\pi$", which would give us 12 solutions, not 18 as he says.]

23. The augmented matrix reduces to

$$\begin{bmatrix} 1 & \lambda - 3 & 0 \\ 0 & (\lambda - 3)^2 - 1 & 0 \end{bmatrix}$$

So, if $\lambda = 2, 4$, then we have non-trivial solutions. [If $\lambda \neq 2, 4$, then we have only the trivial solution.]

Section 1.3

3. (g)

$$\begin{bmatrix} -39 & -21 & -24 \\ 9 & -6 & -15 \\ -33 & -12 & -30 \end{bmatrix}$$

(j) -25

(k) 168

7. (a)

$$\begin{bmatrix} 3 & -2 & 7 \end{bmatrix} \cdot B = \begin{bmatrix} 67 & 41 & 41 \end{bmatrix}$$

(d)

$$B \cdot \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 63 \end{bmatrix}$$

(e)

$$\begin{bmatrix} 0 & 4 & 9 \end{bmatrix} \cdot A = \begin{bmatrix} 24 & 56 & 97 \end{bmatrix}$$

12.(b) [Remember, a product of two matrices is defined only if the number of columns of the first matrix is equal to the number of rows of the second.]

Suppose that B is $r \times s$. Since A is $m \times n$, BA is defined only if $s = m$, in which case BA is $r \times n$. Now, if $A(BA)$ is defined, then $r = n$. Thus, B is $n \times m$.

14.(a)

$$\begin{cases} x_1 - x_2 + 2x_3 = 2 \\ 4x_1 + 3x_2 + 7x_3 = -1 \\ -2x_1 + x_2 + 5x_3 = 4 \end{cases}$$

18.(a) If the i-th row of A is zero, then the i-th row of AB is also zero. We have:

$$\begin{bmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_{i-1} \\ \mathbf{0} \\ \mathbf{r}_{i+1} \\ \vdots \\ \mathbf{r}_m \end{bmatrix} \cdot B = \begin{bmatrix} \mathbf{r}_1 \cdot B \\ \vdots \\ \mathbf{r}_{i-1} \cdot B \\ \mathbf{0} \cdot B \\ \mathbf{r}_{i+1} \cdot B \\ \vdots \\ \mathbf{r}_m \cdot B \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 \cdot B \\ \vdots \\ \mathbf{r}_{i-1} \cdot B \\ \mathbf{0} \\ \mathbf{r}_{i+1} \cdot B \\ \vdots \\ \mathbf{r}_m \cdot B \end{bmatrix}$$

Homework 2

Section 1.4

1. (b)

$$A(BC) = A \cdot \begin{bmatrix} -18 & -62 & -33 \\ 7 & 17 & 22 \\ 11 & -27 & 38 \end{bmatrix} = \begin{bmatrix} -10 & -222 & 26 \\ 83 & -67 & 278 \\ 87 & 33 & 240 \end{bmatrix}$$

and

$$(AB)C = \begin{bmatrix} 28 & -28 & 6 \\ 20 & -31 & 38 \\ 0 & -21 & 36 \end{bmatrix} \cdot C = \begin{bmatrix} -10 & -222 & 26 \\ 83 & -67 & 278 \\ 87 & 33 & 240 \end{bmatrix}.$$

(d)

$$a(B-C)=4\begin{bmatrix} 8 & -1 & -8 \\ -1 & -6 & -2 \\ 1 & -12 & -3 \end{bmatrix} = \begin{bmatrix} 32 & -4 & -32 \\ -4 & -24 & -8 \\ 4 & -48 & -12 \end{bmatrix}$$

and

$$aB-aC=\begin{bmatrix} 32 & -12 & -20 \\ 0 & 4 & 8 \\ 16 & -28 & 24 \end{bmatrix} - \begin{bmatrix} 0 & -8 & 12 \\ 4 & 28 & 16 \\ 12 & 20 & 36 \end{bmatrix} = \begin{bmatrix} 32 & -4 & -32 \\ -4 & -24 & -8 \\ 4 & -48 & -12 \end{bmatrix}.$$

5.(b)

$$\left(B^{\mathrm{T}}\right)^{-1}=\begin{bmatrix} 2 & 4 \\ -3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{5} & -\frac{1}{5} \\ \frac{3}{20} & \frac{1}{10} \end{bmatrix}$$

and

$$\left(B^{-1}\right)^{\mathrm{T}}=\begin{bmatrix} \frac{1}{5} & \frac{3}{20} \\ -\frac{1}{5} & \frac{1}{10} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \frac{1}{5} & -\frac{1}{5} \\ \frac{3}{20} & \frac{1}{10} \end{bmatrix}.$$

6.(a)

$$(AB)^{-1}=\begin{bmatrix} 10 & -5 \\ 18 & -7 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{7}{20} & \frac{1}{4} \\ -\frac{9}{10} & \frac{1}{2} \end{bmatrix}$$

and

$$B^{-1}A^{-1}=\begin{bmatrix} \frac{1}{5} & \frac{3}{20} \\ -\frac{1}{5} & \frac{1}{10} \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} -\frac{7}{20} & \frac{1}{4} \\ -\frac{9}{10} & \frac{1}{2} \end{bmatrix}$$

14. Since

$$A^2-3A+I=0,$$

then,

$$-A^2+3A=I.$$

So, we have,

$$A(3I-A)=3A-A^2=I,$$

and

$$(3I-A)A=3A-A^2=I.$$

Thus, by definition, $A^{-1}=3I-A$. [Note that at first we did not know even if A was invertible, so we could not “multiply by A^{-1}”.]

17. If A is invertible, then we multiplying the equation by A^{-1} on the left we have:

$$A^{-1}AB = A^{-1}0,$$

and so

$$B = 0.$$

21. (a) We have $(BB^{\mathrm{T}})^{\mathrm{T}} = (B^{\mathrm{T}})^{\mathrm{T}}B^{\mathrm{T}} = BB^{\mathrm{T}}$. Hence, $(BB^{\mathrm{T}})^{\mathrm{T}} = BB^{\mathrm{T}}$, and so BB^{T} is symmetric.

Also, $(B+B^{\mathrm{T}})^{\mathrm{T}} = B^{\mathrm{T}}+(B^{\mathrm{T}})^{\mathrm{T}} = B^{\mathrm{T}}+B = B+B^{\mathrm{T}}$. Hence, $(B+B^{\mathrm{T}})^{\mathrm{T}} = B+B^{\mathrm{T}}$, and so $B+B^{\mathrm{T}}$ is symmetric.

(b) Also, $(B-B^{\mathrm{T}})^{\mathrm{T}} = B^{\mathrm{T}}-(B^{\mathrm{T}})^{\mathrm{T}} = B^{\mathrm{T}}-B = -(B-B^{\mathrm{T}})$. Hence, $(B-B^{\mathrm{T}})^{\mathrm{T}} = -(B-B^{\mathrm{T}})$, and so $B-B^{\mathrm{T}}$ is skew-symmetric.

29. (a) If A is invertible and $AB = AC$, then multiplying this equation by A^{-1} on the left, we have $A^{-1}AB = A^{-1}AC$, which gives us $B = C$.

(b) Because matrix A is not invertible [since $ad - bc = 0$].

Section 1.5

5. (a) Switch top and bottom rows.

(a) Multiply top row by 2 and bottom row by -3.

(a) Add add -2 times the top row to the bottom row.

7. (a)

$$\begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\ -1 & 1 & 1 \\ -\frac{1}{2} & \frac{7}{10} & \frac{2}{5} \end{bmatrix}$$

(d)

$$\begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{7}{2} & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

9. (a)

$$\begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{bmatrix}^{-1} = \begin{bmatrix} k_1^{-1} & 0 & 0 & 0 \\ 0 & k_2^{-1} & 0 & 0 \\ 0 & 0 & k_3^{-1} & 0 \\ 0 & 0 & 0 & k_4^{-1} \end{bmatrix}$$

(b)

$$\begin{bmatrix} 0 & 0 & 0 & k_1 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & 0 \\ k_4 & 0 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 0 & k_4^{-1} \\ 0 & 0 & k_3^{-1} & 0 \\ 0 & k_2^{-1} & 0 & 0 \\ k_1^{-1} & 0 & 0 & 0 \end{bmatrix}$$

(c)

$$\begin{bmatrix} k & 0 & 0 & 0 \\ 1 & k & 0 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & k \end{bmatrix}^{-1} = \begin{bmatrix} k^{-1} & 0 & 0 & 0 \\ -k^{-2} & k^{-1} & 0 & 0 \\ k^{-3} & -k^{-2} & k^{-1} & 0 \\ -k^{-4} & k^{-3} & -k^{-2} & k^{-1} \end{bmatrix}$$

15. We see what each one of the three elementary row operations could yield.

(i) *Switch two rows:* This could not happen, since the first two rows remain in their respective places.

(ii) *Multiply a row by a constant:* Clearly we either multiplied the first or second row by 1, in which case the have $a = b = 0$ and $c = 1$, or we multiplied the bottom row by a constant k, in which case we would have $a = b = 0$ and $c = k$. In either case, we would have one [in fact two] zeros in the bottom row.

(iii) *Add a multiple of a row to another row:* We clearly did not add a multiple of any row to either the first or second rows. So, either we added k times the first row to the third, giving $a = k$, $b = 0$, and $c = 1$, or we added k times the second row to the third, giving $a = 0$, $b = k$, and $c = 1$. In either case, we would have one zero in the bottom row.

Section 1.6

2. We have:

$$\begin{bmatrix} 4 & -3 \\ 2 & -5 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{5}{14} & -\frac{3}{14} \\ \frac{1}{7} & -\frac{2}{7} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \frac{5}{14} & -\frac{3}{14} \\ \frac{1}{7} & -\frac{2}{7} \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 9 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix},$$

so $x_1 = x_2 = -3$.

7. We have:

$$\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 2b_1 - 5b_2 \\ -b_1 + 3b_2 \end{bmatrix},$$

so $x_1 = 2b_1 - 5b_2$ and $x_2 = -b_1 + 3b_2$.

16. We have:

$$\begin{bmatrix} 6 & -4 & b_1 \\ 3 & -2 & b_2 \end{bmatrix} \sim \begin{bmatrix} 3 & -2 & b_2 \\ 0 & 0 & b_1 - 2b_2 \end{bmatrix}.$$

So, we need $b_1 - 2b_2 = 0$, or $b_1 = 2b_2$.

17. We have:

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & b_1 \\ 4 & -5 & 8 & b_2 \\ -3 & 3 & -3 & b_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 5 & b_1 \\ 0 & 1 & -4 & (b_2 - 4b_1)/3 \\ 0 & 0 & 0 & b_2 + b_3 - b_1 \end{array}\right]$$

So, we need $b_2 + b_3 - b_1 = 0$, or $b_1 = b_2 + b_3$.

Homework 3

Section 1.7

2. (a) Remember, multiplying by diagonal matrices on the left we multiply *rows*:

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -4 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 4 & -1 \\ 4 & 10 \end{bmatrix}.$$

(b) Remember, multiplying by diagonal matrices on the left we multiply *rows*, multiplying by diagonal matrices on the right we multiply *columns*::

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 4 & -1 & 3 \\ 1 & 2 & 0 \\ -5 & 1 & 2 \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} =$$

$$= \begin{bmatrix} 8 & -2 & 6 \\ -1 & -2 & 0 \\ -20 & 4 & -8 \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -24 & -10 & 12 \\ 3 & -10 & 0 \\ 60 & 20 & -16 \end{bmatrix}$$

5. (a) We use Theorem 1.7.1(c): the product of the elements in the main diagonal is $-1 \cdot 3 \cdot 5 \neq 0$, and hence the matrix is invertible.

(b) We use Theorem 1.7.1(c): the product of the elements in the main diagonal is $0 \cdot 1 \cdot -3 \cdot 5 = 0$, and hence the matrix is *not* invertible.

12. We have:

$$\begin{bmatrix} -1 & 2 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & -4 \end{bmatrix} \cdot \begin{bmatrix} 2 & -8 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 12 & 17 \\ 0 & 2 & 10 \\ 0 & 0 & -12 \end{bmatrix}.$$

So, for these upper triangular matrices, indeed the product is also upper triangular.

13. We have:

$$\begin{bmatrix} -1 & 2 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & -4 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 2 & \frac{1}{4} \\ 0 & 1 & \frac{3}{4} \\ 0 & 0 & -\frac{1}{4} \end{bmatrix}.$$

So, for this upper triangular matrices, indeed the inverse is also upper triangular.

15. (a) A symmetric means $A^{\mathrm{T}} = A$. So,

$$(A^2)^{\mathrm{T}} = (A \cdot A)^{\mathrm{T}} = A^{\mathrm{T}} \cdot A^{\mathrm{T}} = A \cdot A = A^2.$$

[Note that the second equality holds since $(A \cdot B)^{\mathrm{T}} = B^{\mathrm{T}} \cdot A^{\mathrm{T}}$.] Thus $(A^2)^{\mathrm{T}} = A^2$, i.e., A^2 is symmetric.

(b) We have:

$$\begin{aligned} (2A^2 - 3A + I)^{\mathrm{T}} &= (2A^2)^{\mathrm{T}} + (-3A)^{\mathrm{T}} + I^{\mathrm{T}} && \text{[since } (A+B)^{\mathrm{T}} = A^{\mathrm{T}} + B^{\mathrm{T}}\text{]} \\ &= 2(A^2)^{\mathrm{T}} - 3A^{\mathrm{T}} + I && \text{[since } (kA)^{\mathrm{T}} = kA^{\mathrm{T}}\text{, and } I \text{ is sym.]} \\ &= 2A^2 - 3A + I && \text{[since } A \text{ is sym., and using (a)]} \end{aligned}$$

Thus, $2A^2 - 3A + I$ is symmetric.

18. We have

$$A^{\mathrm{T}} = (A^{\mathrm{T}}A)^{\mathrm{T}} = A^{\mathrm{T}}(A^{\mathrm{T}})^{\mathrm{T}} = A^{\mathrm{T}}A = A,$$

and hence A is symmetric, i.e., $A = A^{\mathrm{T}}$. But then, $A = A^{\mathrm{T}}A = A\,A = A^2$.

Section 2.1

7. Along the first column:

$$\begin{aligned}\begin{vmatrix} 1 & k & k^2 \\ 1 & k & k^2 \\ 1 & k & k^2 \end{vmatrix} &= 1 \cdot \begin{vmatrix} k & k^2 \\ k & k^2 \end{vmatrix} - 1 \cdot \begin{vmatrix} k & k^2 \\ k & k^2 \end{vmatrix} + 1 \cdot \begin{vmatrix} k & k^2 \\ k & k^2 \end{vmatrix} \\ &= (k^3 - k^3) - (k^3 - k^3) + (k^3 - k^3) \\ &= 0\end{aligned}$$

9. $\det A = -240$.

12. We have $\det A = -6$,:

$$\operatorname{adj}(A) = \begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix} \qquad \text{and} \qquad A^{-1} = \begin{bmatrix} 2 & 0 & \frac{3}{2} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -1 & 0 & -1 \end{bmatrix}.$$

17. We have

$$\det A = \begin{vmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix} = -132$$

and

$$\begin{vmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix} = -24, \qquad \begin{vmatrix} 4 & 2 & 0 \\ 11 & 3 & 2 \\ 1 & 1 & 2 \end{vmatrix} = -36, \qquad = \begin{vmatrix} 4 & 5 & 2 \\ 11 & 1 & 3 \\ 1 & 5 & 1 \end{vmatrix} = 12.$$

By Cramer's Rule,

$$x = \frac{-24}{-132} = \frac{3}{11}, \qquad y = \frac{-36}{-132} = \frac{2}{11}, \qquad z = \frac{12}{-132} = \frac{-1}{11}.$$

Section 2.2

2.(b) We have:

$$\begin{vmatrix} 2 & -1 & 3 \\ 1 & 2 & 4 \\ 5 & -3 & 6 \end{vmatrix} = 24 - 20 - 9 - (30 - 24 - 6) = -5$$

and

$$\begin{vmatrix} 2 & 1 & 5 \\ -1 & 2 & -3 \\ 3 & 4 & 6 \end{vmatrix} = 24 - 9 - 20 - (30 - 24 - 6) = -5.$$

3. (a) -5 [multiply a row of I by -5].

(b) -1 [switch second and third rows of I].

(c) 1 [add -9 times the fourth row of I to the second row].

9. We have:

$$\begin{vmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 3 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 1 & 0 \\ 2 & 1 & 3 & 1 \end{vmatrix} =$$

$$\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -3 & -6 \\ 0 & 0 & -1 & -4 \end{vmatrix} = -3 \cdot \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & -4 \end{vmatrix}$$

$$= -3 \cdot \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -2 \end{vmatrix} = (-3)(-2) \cdot \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 6$$

14.(a) The easiest way would be to switch the top and bottom rows [which adds a negative sign to the determinant], thus obtaining an upper triangular matrix, of which we can compute the determinant by multiplying the main diagonal, which yields the desired result. But, since the problem asks to compute as in the proof of Theorem 2.1.3, we compute it by going through the "first rows":

$$\begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{13} \cdot \begin{vmatrix} 0 & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{13} \cdot (-a_{22}) \cdot a_{31}.$$

Section 2.3

2. We have $\det A = 10$, $\det B = -17$, and

$$\det(A \cdot B) = \begin{vmatrix} 9 & -1 & 8 \\ 31 & 1 & 17 \\ 10 & 0 & 2 \end{vmatrix} = -170.$$

5. (a) $\det(3A) = 3^3 \det A = -189.$

(b) $\det(A^{-1}) = \dfrac{1}{\det A} = -\dfrac{1}{7}.$

(c) $\det(2A^{-1}) = 2^3 \det(A^{-1}) = -\dfrac{8}{7}.$

(d) $\det((2A)^{-1}) = \dfrac{1}{\det(2A)} = \dfrac{1}{2^3 \det A} = -\dfrac{1}{56}.$

(e) $\begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix} = -\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} = -\det(A^{\mathrm{T}}) = -\det A = 7.$

15. (a) (i) $\lambda^2 - 2\lambda - 3 = 0$

(ii) $\lambda = 3, -1.$

(iii) For $\lambda = 3$, we have $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}$, for all $t \in \mathbb{R}$.

For $\lambda = -1$, we have $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ -t \end{bmatrix}$, for all $t \in \mathbb{R}$.

(b) (i) $\lambda^2 - 5\lambda - 6 = 0$

(ii) $\lambda = 6, -1.$

(iii) For $\lambda = 6$, we have $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ 4t/3 \end{bmatrix}$, for all $t \in \mathbb{R}$.

For $\lambda = -1$, we have $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ -t \end{bmatrix}$, for all $t \in \mathbb{R}$.

Homework 4

Section 3.1

7. We have that $6\mathbf{x} = 2\mathbf{u} - \mathbf{v} - \mathbf{w}$, or

$$\mathbf{x} = \frac{1}{3}\mathbf{u} - \frac{1}{6}\mathbf{v} - \frac{1}{6}\mathbf{w} = \left(-\frac{8}{3}, \frac{1}{2}, \frac{7}{3}\right).$$

9. The resulting system of equations is:

$$\begin{bmatrix} -2 & -3 & 1 \\ 9 & 2 & 7 \\ 6 & 1 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix}.$$

Computing the echelon form of the augmented matrix, we get:

$$\left[\begin{array}{ccc|c} -2 & -3 & 1 & 0 \\ 9 & 2 & 7 & 5 \\ 6 & 1 & 5 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3/2 & -1/2 & 0 \\ 0 & 1 & -1 & -10/23 \\ 0 & 0 & 0 & 7/46 \end{array}\right].$$

To, the systems has no solution, i.e., there are no c_1, c_2, and c_3 that would make the given equation true.

11. (a) We have $\overrightarrow{PQ} = Q - P = (5, -7, 3)$. Then the midpoint is $P + \frac{1}{2}\overrightarrow{PQ} = \left(\frac{9}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$. [If you don't see why, draw a picture!]

(b) The point three fourths of the way from P to Q is $P + \frac{3}{4}\overrightarrow{PQ} = \left(\frac{23}{4}, -\frac{9}{4}, \frac{1}{4}\right)$.

13. If $M = (4, 0, -6)$, then

$$Q = P + 2 \cdot \overrightarrow{PM} = (1, 3, 7) + 2 \cdot (4 - 1, 0 - 3, -6 - 7) = (7, -3, -19).$$

Section 3.2

2. (a) $d(P_1, P_2) = \left\|\overrightarrow{P_1P_2}\right\| = \|P_2 - P_1\| = \|(2, 3)\| = \sqrt{2^2 + 3^2} = \sqrt{13}$.

(c) $d(P_1, P_2) = \|(-14, 3, -2)\| = \sqrt{(-14)^2 + 3^2 + (-2)^2} = \sqrt{209}$.

4. Remember that in a triangle whose sides measure a, b, and c, we always have $c < a + b$. [And since c was any side, we also have $b < a + c$ and $a < b + c$.]

If $\mathbf{v}$ and $\mathbf{w}$ are *not* parallel [i.e., on the same line through the origin], then $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{v} - \mathbf{w}$ make a triangle. Thus $\|\mathbf{v} - \mathbf{w}\| < \|\mathbf{v}\| + \|\mathbf{w}\| = 5$.

Also, if we still assume that $\mathbf{v}$ and $\mathbf{w}$ are not parallel, we have that $\|\mathbf{w}\| < \|\mathbf{v}\| + \|\mathbf{v} - \mathbf{w}\|$, which implies that $\|\mathbf{v} - \mathbf{w}\| > \|\mathbf{w}\| - \|\mathbf{v}\| = 1$.

[So, the two paragraphs above, tells us that in the non-parallel case, we have

$$1 < \|\mathbf{v} + \mathbf{w}\| < 5.]$$

But, if $\mathbf{v}$ and $\mathbf{w}$ are parallel, but in opposite directions, then $\|\mathbf{v}-\mathbf{w}\| = \|\mathbf{v}\|+\|\mathbf{w}\| = 5$ [which is larger than in the non-parallel case], and if they have the same direction, then $\|\mathbf{v}-\mathbf{w}\| = 1$ [which is smaller than in the non-parallel case]. [Draw pictures!!]

Thus, the maximal value of $\|\mathbf{v}-\mathbf{w}\|$ is 5 and occurs when the vectors are parallel and in opposite direction, and the minimal value for $\|\mathbf{v}-\mathbf{w}\|$ is 1 and occurs when the vectors are parallel and with the same direction.

5.(a) We have:

$$k\mathbf{u} + l\mathbf{v} = k(2,0,4) + l(1,3,-6) = (2k+l, 3l, 4k-6l).$$

So, if $k\mathbf{u} + l\mathbf{v} = (5,9,-14)$, then

$$\begin{aligned} 2k + \ \ l &= \ \ 5 \\ 3l &= \ \ 9 \\ 4k - 6l &= -14 \end{aligned}$$

The second equation gives us that $l = 3$. Substituting on the first, we get that $k = 1$. Now, we need to check that the third equation holds for those values, but indeed, $4 \cdot 1 - 6 \cdot 3 = -14$. Hence, we have $k = 1$ and $l = 3$.

[**Note:** Be careful to verify that all equations are satisfied! For instance, if we have $k\mathbf{u}+l\mathbf{v} = (5,9,-13)$, instead, there would be no k and l that would make this equation true.]

Section 3.3

2. (a)

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{-11}{\sqrt{13}\,\sqrt{74}} = -\frac{11}{\sqrt{962}}.$$

(b)

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{0}{\sqrt{17}\,\sqrt{66}} = 0.$$

4. (a)

$$\operatorname{proj}_{\mathbf{a}} \mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\,\mathbf{a} = 0\cdot\mathbf{a} = \mathbf{0}.$$

[This means that the vector $\mathbf{u}$ is perpendicular to $\mathbf{a}$, so the projection is has length 0. Draw a picture.]

(b)

$$\operatorname{proj}_{\mathbf{a}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2} \mathbf{a} = \left(-\frac{16}{13}, 0, -\frac{80}{13}\right).$$

9. (a) 102.

(b) $125\sqrt{2}$.

(c) 170.

(d) 170.

12. We have $\overrightarrow{AB} = (1, 3, -2)$, $\overrightarrow{BC} = (4, -2, 1)$, and $\overrightarrow{CA} = (-5, -1, 3)$. So, either one can see that $\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$, and so the vertex B is the vertex with the right angle, or one can realize that

$$\left\|\overrightarrow{CA}\right\|^2 = 35 = 14 + 21 = \left\|\overrightarrow{AB}\right\|^2 + \left\|\overrightarrow{BC}\right\|^2,$$

and so [by Pythagoras] $\overrightarrow{CA}$ is the hypotenuse, which means that the right angle has vertex B.

Section 4.1

7. We have that $\|k\,\mathbf{v}\| = |k|\,\|\mathbf{v}\|$. Thus:

$$\left\|\frac{1}{\|\mathbf{v}\|}\mathbf{v}\right\| = \left|\frac{1}{\|\mathbf{v}\|}\right| \|\mathbf{v}\| = \frac{\|\mathbf{v}\|}{\|\mathbf{v}\|} = 1.$$

9.(d) 11.

11.(c) $\sqrt{3^2 + (-4)^2 + (-5)^2 + (-3)^2} = \sqrt{59}$.

16. Let $\mathbf{x} = (x_1, x_2, x_3, x_4)$ be a vector orthogonal to $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. So, we have that its dot product with this three vectors must be equal to zero, giving us the following homogeneous system:

$$\begin{bmatrix} 2 & 1 & -4 & 0 \\ -1 & -1 & 2 & 2 \\ 3 & 2 & 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Putting the matrix of *coefficients* [I'm not using the augmented matrix since the system is homogeneous] in reduced echelon form, we get:

$$\begin{bmatrix} 1 & 0 & 0 & \frac{34}{11} \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & \frac{6}{11} \end{bmatrix},$$

giving us solutions:

$$x_1 = -34t, \quad x_2 = 44t, \quad x_3 = -6t, \quad x_4 = 11t,$$

for all $t \in \mathbb{R}$. [Note, I took $x_4 = 11t$ instead of $x_4 = t$ to just to avoid the fractions. Observe that we can always assign to a free variable a non-zero multiple of the parameter t instead of t itself. But, if one would use $x_1 = -34t/11$, $x_2 = 4t/11$, $x_3 = -6t/11$, $x_4 = t$, it would also be correct.]

So, those are *all* vectors orthogonal to the three given ones. We want to find the ones of length one, so:

$$\|\mathbf{x}\| = \sqrt{(34t)^2 + (44t)^2 + (6t)^2 + (11t)^2} = 57\,|t| = 1.$$

Therefore, $t = \pm 1/57$, giving us the two solutions:

$$x_1 = -34/57, \quad x_2 = 44/57, \quad x_3 = -6/57, \quad x_4 = 11/57,$$

and

$$x_1 = 34/57, \quad x_2 = -44/57, \quad x_3 = 6/57, \quad x_4 = -11/57.$$

17.(d) We have $\mathbf{u} \cdot \mathbf{v} = 5$, $\|\mathbf{u}\| = \sqrt{9} = 3$, and $\|\mathbf{v}\| = \sqrt{4} = 2$. And, indeed, as the Cauchy-Schwarz inequality states:

$$|\mathbf{u} \cdot \mathbf{v}| = |5| = 5 \leq \|\mathbf{u}\|\,\|\mathbf{v}\| = 3 \cdot 2 = 6.$$

18. We have:

$$A\mathbf{u} = \begin{bmatrix} 25 \\ -1 \\ 6 \end{bmatrix}, \qquad A^{\mathrm{T}}\mathbf{v} = \begin{bmatrix} -14 \\ 10 \\ -12 \end{bmatrix}.$$

Thus $(A\mathbf{u}) \cdot \mathbf{v} = -26 = \mathbf{u} \cdot (A^{\mathrm{T}}\mathbf{v})$, which is formula (8).

Also,

$$A\mathbf{v} = \begin{bmatrix} -12 \\ 2 \\ -16 \end{bmatrix}, \qquad A^{\mathrm{T}}\mathbf{u} = \begin{bmatrix} 32 \\ -10 \\ 11 \end{bmatrix}$$

So, $\mathbf{u} \cdot (A\mathbf{v}) = -64 = (A^{\mathrm{T}}\mathbf{u}) \cdot \mathbf{v}$, which is formula (9).

20. We have:

$$\mathbf{u} \cdot \mathbf{v} = \frac{1}{4}\|\mathbf{u}+\mathbf{v}\| - \frac{1}{4}\|\mathbf{u}-\mathbf{v}\| = \frac{1}{4} - \frac{5}{4} = -1.$$

Section 4.2

12. (a) $\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$

(b) $\begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}$

(c) $\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

16. (a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ [Careful with the order!!! Rotation first [on the right], then reflection [on the left].]

(b) $\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1/2 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$

Homework 5

Section 4.2

20. (a)

$$[T_2 \circ T_1] = [T_2] \cdot [T_1] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$[T_1 \circ T_2] = [T_1] \cdot [T_2] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

So, $T_1 \circ T_2 = T_2 \circ T_1$.

(b) We have that $T_1 \circ T_2$ and $T_2 \circ T_1$ are both rotations of $\theta_1 + \theta_2$, so are equal. If you want to see it algebraically:

$$\begin{aligned} [T_2 \circ T_1] = [T_2] \cdot [T_1] &= \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta_2\cos\theta_1 - \sin\theta_2\sin\theta_1 & -(\sin\theta_2\sin\theta_1 + \cos\theta_2\cos\theta_1) \\ \sin\theta_2\sin\theta_1 + \cos\theta_2\cos\theta_1 & \cos\theta_2\cos\theta_1 - \sin\theta_2\sin\theta_1 \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} [T_1 \circ T_2] = [T_1] \cdot [T_2] &= \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & -(\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2) \\ \sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2 & \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}. \end{aligned}$$

So, indeed, $T_1 \circ T_2 = T_2 \circ T_1$.

(c) We have:

$$[T_2 \circ T_1] = [T_2] \cdot [T_1] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \end{bmatrix},$$

and

$$[T_1 \circ T_2] = [T_1] \cdot [T_2] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ 0 & 0 \end{bmatrix}.$$

Thus, if $\sin\theta \neq 0$, then $T_1 \circ T_2 \neq T_2 \circ T_1$. So, they are equal in, and only if, θ is an integer multiple of π.

Section 4.3

2. (b) Matrix: $\begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix}$. Since the determinant is $17 \neq 0$, the operator is one-to-one.

(c) Matrix: $\begin{bmatrix} -1 & 3 & 2 \\ 2 & 0 & 4 \\ 1 & 3 & 6 \end{bmatrix}$. Since the determinant is zero, the operator is *not* one-to-one.

4. One could compute the determinant of the matrix $\begin{bmatrix} 1 & -2 & 1 \\ 5 & -1 & 3 \\ 4 & 1 & 2 \end{bmatrix}$. Since it's zero, we can conclude that it's not onto [or one-to-one]. On the other hand, since we need to find a vector not in the range, this is enough to show that the operator is not onto, and hence computing this determinant would be a waste of time, unless you want to double check it.

So, we just go ahead and find the vector not in the range. A vector $\mathbf{w} = (w_1, w_2, w_2)$ is in the range, if there is an $\mathbf{x} = (x_1, x_2, x_3)$ such that

$$\begin{bmatrix} 1 & -2 & 1 \\ 5 & -1 & 3 \\ 4 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

Solving the system we get:

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & w_1 \\ 5 & -1 & 3 & w_2 \\ 4 & 1 & 2 & w_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & w_1 \\ 0 & 9 & -2 & w_2 - 5w_1 \\ 0 & 9 & -2 & w_3 - 4w_1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & w_1 \\ 0 & 9 & -2 & w_2 - 5w_1 \\ 0 & 0 & 0 & w_3 - w_2 + w_1 \end{array}\right].$$

So, the system has no solution [i.e., $\mathbf{w}$ is not in the range] if, and only if, $w_3 - w_2 + w_1 \neq 0$ [by looking at the bottom row]. So, we just need to choose $w_1, w_2, w_3 \in \mathbb{R}$ such that $w_3 - w_2 + w_1 \neq 0$. For instance $w_1 = 1$, $w_2 = 0$, and $w_3 = 0$. So, $\mathbf{w} = (1, 0, 0)$ is not in the range. Since there is a vector not in the range, the operator is not onto.

6.(b) The matrix associate to the linear transformation is $\begin{bmatrix} 1 & -3 & 4 \\ -1 & 1 & 1 \\ 0 & -2 & 5 \end{bmatrix}$, which has determinant equal to 0, the transformation is not one-to-one.

[**Note:** If it were, the inverse operator T^{-1} would be given by the inverse of its matrix, and to find $T^{-1}(w_1, w_2, w_3)$ we would multiply the vector (w_1, w_2, w_3) [as a column matrix] by the inverse matrix [on the right].]

8. (a) Let $\mathbf{v} = (x_1, y_1)$, $\mathbf{w} = (x_2, y_2)$ and $k \in \mathbb{R}$. We have:

$$\begin{aligned} T(\mathbf{v} + \mathbf{w}) &= T(x_1 + x_2, y_1 + y_2) \\ &= (2(x_1 + x_2), y_1 + y_2) \\ &= (2x_1 + 2x_2, y_1 + y_2) \\ &= (2x_1, y_1) + (2x_2, y_2) \\ &= T(\mathbf{v}) + T(\mathbf{w}). \end{aligned}$$

Also,

$$\begin{aligned} T(k\mathbf{v}) &= T(kx_1, ky_1) \\ &= (2(kx_1), ky_1) \\ &= k(2x_1, y_1) \\ &= kT(\mathbf{v}). \end{aligned}$$

Thus, T is linear.

(b) Let $\mathbf{v} = (x_1, y_1)$, $\mathbf{w} = (x_2, y_2)$ and $k \in \mathbb{R}$. We have:

$$\begin{aligned} T(\mathbf{v} + \mathbf{w}) &= T(x_1 + x_2, y_1 + y_2) \\ &= ((x_1 + x + 2)^2, y_1 + y_2) \\ &= (x_1^2 + 2x_1x_2 + x_2^2, y_1 + y_2) \\ &= (x_1^2, y_1) + (2x_2^2, y_2) + (2x_1x_2, 0) \\ &= T(\mathbf{v}) + T(\mathbf{w}) + (2x_1x_2, 0). \end{aligned}$$

So, $T(\mathbf{v} + \mathbf{w}) \neq T(\mathbf{v}) + T(\mathbf{w})$ unless $2x_1x_2 = 0$ [i.e., $x_1 = 0$ or $x_2 = 0$]. For instance, $T((1, 0) + (1, 1)) \neq T(1, 0) + T(1, 1)$. So, T is not linear.

[**Note:** We do not have check the other property, since failing one already tells us that T is not linear. But, here is how you would check the second one:

$$\begin{aligned} T(k\mathbf{v}) &= T((kx_1)^2, ky_1) \\ &= (k^2x_1^2, ky_1) \\ &= k(kx_1^2, y_1). \end{aligned}$$

So, unless $(kx_1^2, y_1) = T(\mathbf{v}) = (x_1^2, y_1)$ [i.e., $x_1 = 0$, or $k = 1$], we have that $T(k\mathbf{v}) \neq kT(\mathbf{v})$. For instance, $T(2(1,0)) \neq 2T(1,0)$.]

(c) Let $\mathbf{v} = (x_1, y_1)$, $\mathbf{w} = (x_2, y_2)$ and $k \in \mathbb{R}$. We have:

$$\begin{aligned} T(\mathbf{v}+\mathbf{w}) &= T(x_1+x_2, y_1+y_2) \\ &= (-(y_1+y_2), x_1+x_2) \\ &= (-y_1 + -y_2, x_1+x_2) \\ &= (-y_1, x_1) + (-y_2, x_2) \\ &= T(\mathbf{v}) + T(\mathbf{w}). \end{aligned}$$

Also,

$$\begin{aligned} T(k\mathbf{v}) &= T(kx_1, ky_1) \\ &= (-(ky_1), kx_1) \\ &= k(-y_1, x_1) \\ &= kT(\mathbf{v}). \end{aligned}$$

Thus, T is linear.

(d) Let $\mathbf{v} = (x_1, y_1)$, $\mathbf{w} = (x_2, y_2)$ and $k \in \mathbb{R}$. We have:

$$\begin{aligned} T(\mathbf{v}+\mathbf{w}) &= T(x_1+x_2, y_1+y_2) \\ &= ((x_1+x_2), 0) \\ &= (x_1, 0) + (x_2, 0) \\ &= T(\mathbf{v}) + T(\mathbf{w}). \end{aligned}$$

Also,

$$\begin{aligned} T(k\mathbf{v}) &= T(kx_1, ky_1) \\ &= ((kx_1), 0) \\ &= k(x_1, 0) \\ &= kT(\mathbf{v}). \end{aligned}$$

Thus, T is linear.

11. (a) Let $\mathbf{v} = (x_1, y_1, z_1)$, $\mathbf{w} = (x_2, y_2, z_2)$ and $k \in \mathbb{R}$. We have:

$$\begin{aligned} T(\mathbf{v}+\mathbf{w}) &= T(x_1+x_2, y_1+y_2, z_1+z_2) \\ &= (0,0) \\ &= (0,0)+(0,0) \\ &= T(\mathbf{v}) + T(\mathbf{w}). \end{aligned}$$

Also,

$$\begin{aligned} T(k\mathbf{v}) &= T(kx_1, ky_1, kz_1) \\ &= (0,0) \\ &= k(0,0) \\ &= kT(\mathbf{v}). \end{aligned}$$

Thus, T is linear.

(a) Let $\mathbf{v} = (x_1, y_1, z_1)$, $\mathbf{w} = (x_2, y_2, z_2)$ and $k \in \mathbb{R}$. We have:

$$\begin{aligned} T(\mathbf{v}+\mathbf{w}) &= T(x_1+x_2, y_1+y_2, z_1+z_2) \\ &= (3(x_1+x_2) - 4(y_1+y_2), 2(x_1+x_2) - 5(z_1+z_2)) \\ &= (3x_1 - 4y_1 + 3x_2 - 4y_2, 2x_1 - 5z_1 2 + x_2 - 5z_2) \\ &= (3x_1 - 4y_1, 2x_1 - 5z_1) + (3x_2 - 4y_2, 2x_2 - 5z_2) \\ &= T(\mathbf{v}) + T(\mathbf{w}). \end{aligned}$$

Also,

$$\begin{aligned} T(k\mathbf{v}) &= T(kx_1, ky_1, kz_1) \\ &= (3(kx_1) - 4(ky_1), 2(kx_1) - 5(kz_1)) \\ &= k(3x_1 - 4y_1, 2x_1 - 5z_1) \\ &= kT(\mathbf{v}). \end{aligned}$$

Thus, T is linear.

15. The idea of this problem is *not* to compute the results by matrix multiplication, but to use Theorem 4.3.3. The columns of the matrix are $T(\mathbf{e}_1)$, $T(\mathbf{e}_2)$, and $T(\mathbf{e}_3)$ respectively.

(a) $T(\mathbf{e}_1) = (-1, 2, 4)$ [first column], $T(\mathbf{e}_2) = (3, 1, 5)$ [second column], $T(\mathbf{e}_3) = (0, 2, -3)$ [third column].

(b) $T(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = T(\mathbf{e}_1) + T(\mathbf{e}_2) + T(\mathbf{e}_3)$ [since T is linear], and so it results in the sum of the three columns: $(2, 5, 6)$.

(c) $T(7\mathbf{e}_1) = 7T(\mathbf{e}_3)$ [since T is linear], and so it results in seven times the sum of the third column: $(0, 14, -21)$.

22. Remember that a linear transformation satisfies: $T(\mathbf{v}+\mathbf{w}) = T(\mathbf{v})+T(\mathbf{w})$ and $T(k\mathbf{v}) = kT(\mathbf{v})$. Then:

$$T(\mathbf{0}) = T(0 \cdot \mathbf{0}) = 0 \cdot T(\mathbf{0}) = 0.$$

[Alternatively, you could also have done:

$$T(\mathbf{0}) = T(\mathbf{0} + \mathbf{0}) = T(\mathbf{0}) + T(\mathbf{0}).$$

Then, subtracting $T(\mathbf{0})$ from both sides also gives us $T(\mathbf{0}) = \mathbf{0}$.]

Homework 6

Section 5.1

6. (1) If $(x, y), (z, w) \in V$ [so $x, z \geq 0$], then $(x, y) + (z, w) = (x + z, y + w)$ is also in V, since $x, y \geq 0$ implies that $x + y \geq 0$. **OK.**

(2), (3) These hold for all vectors in $\mathbb{R}^2$ [since $\mathbb{R}^2$ is a vector space], so, in particular, they hold for vectors with positive first coordinate. **OK.**

(4) $\mathbb{R}^2$ has a zero vector, namely $\mathbf{0} = (0, 0)$. Since $0 \geq 0$, then $\mathbf{0} \in V$, and this vector is also the zero vector of V. **OK.**

(5) Note that $(1, 0) \in V$, and the only vector $\mathbf{v}$ in $\mathbb{R}^2$ such that $\mathbf{v} + (1, 0) = \mathbf{0} = (0, 0)$ is $(-1, 0)$. But since $-1 < 0$, the vector $(-1, 0) \notin V$. So, there is no $-(1, 0) \in V$. **Fails.**

(6) Again, $(1, 0) \in V$, and take $-1 \in \mathbb{R}$. Then $-1 \cdot (1, 0) = (-1, 0) \notin V$. **Fails.**

(7) to (10) These all hold for all vectors in $\mathbb{R}^2$, so, in particular, it holds for all vectors in V. **OK.**

Note: It's very important that you realize why we can say things like "these all hold for all vectors in $\mathbb{R}^2$, so, in particular, it holds for all vectors in V" in cases (2), (3), and (7) to (10), but not in the other cases. Can you see why? This is because:

(a) the addition and scalar multiplication of V are the same as the ones in $\mathbb{R}^2$;

(b) those items [(2),(3), and (7) to (10)] do not deal with *existence* ["there exists"] or *containment* ["is in V"], but merely *algebraic properties*, which are then inherited by any subset.

11. By Example 4, real valued functions form a vector space. As above, *since we are using the same addition and scalar multiplication as the Example 4*, the properties that can fail are (1), (4), (5), and (6).

(1) Let $f(x), g(x) \in V$. Then $f(0), g(0) = 0$. Then, $(f+g)(x) = f(x) + g(x)$, and so $(f+g)(0) = f(0) + g(0) = 0 + 0 = 0$. Thus, $(f+g)(x) \in V$. **OK.**

(4) The zero vector of the vector space of real valued functions defined for all reals is function constant equal to zero $f \equiv 0$ [i.e., $f(x) = 0$ for all $x \in \mathbb{R}$]. Then, in particular, $f(0) = 0$, and hence this function is also in V, and is the zero vector of V. **OK.**

(5) If $f(x) \in V$, then $f(0) = 0$. But then, $(-f)(x) = -f(x)$ and so $(-f)(0) = -f(0) = -0 = 0$. Hence, $(-f)(x) \in V$. **OK.**

(6) If $f(x) \in V$, then $f(0) = 0$. Let $k \in \mathbb{R}$. Then, $(kf)(x) = k \cdot f(x)$, and so $(kf)(0) = k \cdot f(0) = k \cdot 0 = 0$. Thus, $(kf)(x) \in V$. **OK.**

Therefore, V is a vector space.

15. Note that now we are not using the same addition and scalar multiplication as $\mathbb{R}$, so we cannot take the same shortcut as the previous exercises. So, here $V = (0, \infty)$.

To differentiate the sum of real numbers with the sum in V [which is *defined* as the product], and to differentiate the product of real numbers from the scalar multiplication in V [which is defined as exponentiation], I will use "$\oplus$" and "$\odot$" for the operations in V, i.e., if $x, y \in V$ and $k \in \mathbb{R}$, we have

$$x \oplus y = xy \qquad \text{and} \qquad k \odot x = x^k.$$

(1) If $x, y \in V$, then their sum is $x \oplus y = xy$. Since x and y are positive real numbers, then so is xy. Thus, if $x, y \in V$, then $x \oplus y \in V$. **OK.**

(2) Let $x, y \in V$. Then, $x \oplus y = xy = yx = y \oplus x$. **OK.**

(3) Let $x, y, z \in V$. Then, $x \oplus (y \oplus z) = x \oplus yz = xyz = xy \oplus z = (x \oplus y) \oplus z$. **OK.**

(4) Note that $0 \notin V$, but 0 is not what we want as the zero of the vector space. We want $\mathbf{0}$ such that $\mathbf{0} \oplus x = x$, for all $x \in V$, i.e., $\mathbf{0} \cdot x = x$, for all $x \in (0, \infty)$. Clearly, what we want then is $\mathbf{0} = 1$. Since $1 \in V$, and $1 \oplus x = x$ for all $x \in V$, we have that 1 is our zero vector in this case. **OK.**

(5) Again, if $x \in V = (0, \infty)$, clearly $-x < 0$, and so it would not be in V. But, as in (4), this is not what we want. We want that given x, there is $y \in z$ such that $x \oplus y = \mathbf{0}$. But this means $xy = 1$. So, the equivalent of "$-x$" for this vector space is x^{-1}. Note that since $x \in V = (0, \infty)$, then $x^{-1} \in (0, \infty) = V$, and $x \oplus x^{-1} = 1 = \mathbf{0}$. **OK.**

(6) Since $V = (0, \infty)$, for all $x \in V$ and $k \in \mathbb{R}$, we have that $k \odot x = x^k \in (0, \infty) = V$. [Note it would be false if we allowed negative numbers are base: $(-1)^{1/2} = \sqrt{-1} \notin \mathbb{R}$.] **OK.**

(7) Let $x, y \in V$ and $k \in \mathbb{R}$. Then $k \odot (x \oplus y) = k \odot (xy) = (xy)^k = x^k y^k = x^k \oplus y^k = (k \odot x) \oplus (k \odot y)$. **OK.**

(8) Let $x \in V$ and $k, l \in \mathbb{R}$. Then $(k+l) \odot x = x^{k+l} = x^k x^l = x^k \oplus x^l = (k \odot x) \oplus (l \odot x)$. **OK.**

(9) Let $x \in V$ and $k, l \in \mathbb{R}$. Then $(k \cdot l) \odot x = x^{k \cdot l} = (x^l)^k = (l \odot x)^k = k \odot (l \odot x)$. **OK.**

(10) Let $x \in V$. Then $1 \odot x = x^1 = x$. **OK.**

Thus, V is a vector space.

18. Again, since we are changing the usual operations of matrices [more precisely, of $M_{2,2}$], we cannot take the shortcut and need to check all properties. As, as shortcut, let:

$$\mathbf{u} = \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} c & 1 \\ 1 & d \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} e & 1 \\ 1 & f \end{bmatrix}.$$

[Note that these are *generic* elements of V.]

(1) If $\mathbf{u}, \mathbf{w} \in V$, then

$$\mathbf{v} + \mathbf{w} = \begin{bmatrix} (a+c) & 1 \\ 1 & (b+d) \end{bmatrix} \in V.$$

OK.

(2) We have that

$$\mathbf{u}+\mathbf{v}=\begin{bmatrix}(a+c) & 1\\ 1 & (b+d)\end{bmatrix}=\begin{bmatrix}(c+a) & 1\\ 1 & (d+b)\end{bmatrix}=\mathbf{v}+\mathbf{u}.$$

OK.

(3) We have that

$$\mathbf{u}+(\mathbf{v}+\mathbf{w})=\begin{bmatrix}(a+c+e) & 1\\ 1 & (b+d+f)\end{bmatrix}=(\mathbf{u}+\mathbf{v})+\mathbf{w}.$$

OK.

(4) Let $\mathbf{0}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$. Then, $\mathbf{0}\in V$ and

$$\mathbf{u}+\mathbf{0}=\begin{bmatrix}(a+0) & 1\\ 1 & (b+0)\end{bmatrix}=\mathbf{u}.$$

OK.

(5) We have that if $\mathbf{u}\in V$, then

$$\mathbf{u}+\begin{bmatrix}-a & 1\\ 1 & -b\end{bmatrix}=\mathbf{0}.$$

Thus, $-\mathbf{u}=\begin{bmatrix}-a & 1\\ 1 & -b\end{bmatrix}\in V$. **OK.**

(6) If $k\in\mathbb{R}$, then

$$k\mathbf{u}=\begin{bmatrix}ka & 1\\ 1 & kb\end{bmatrix}\in V.$$

OK.

(7) If $k\in\mathbb{R}$, then

$$\begin{aligned}k(\mathbf{u}+\mathbf{v})&=k\begin{bmatrix}a+c & 1\\ 1 & b+d\end{bmatrix}=\begin{bmatrix}k(a+c) & 1\\ 1 & k(b+d)\end{bmatrix}\\ &=\begin{bmatrix}ka+kc & 1\\ 1 & kb+kd\end{bmatrix}=\begin{bmatrix}ka & 1\\ 1 & kb\end{bmatrix}+\begin{bmatrix}kc & 1\\ 1 & kd\end{bmatrix}=k\mathbf{u}+k\mathbf{v}.\end{aligned}$$

OK.

(8) If $k, l \in \mathbb{R}$, then

$$(k+l)\mathbf{u} = \begin{bmatrix} (k+l)a & 1 \\ 1 & (k+l)b \end{bmatrix} = \begin{bmatrix} ka+la & 1 \\ 1 & kb+lb \end{bmatrix}$$

$$= \begin{bmatrix} ka & 1 \\ 1 & kb \end{bmatrix} + \begin{bmatrix} la & 1 \\ 1 & lb \end{bmatrix} = k\mathbf{u} + l\mathbf{u}.$$

OK.

(9) If $k, l \in \mathbb{R}$, then

$$(kl)\mathbf{u} = \begin{bmatrix} (kl)a & 1 \\ 1 & (kl)b \end{bmatrix} = k \begin{bmatrix} la & 1 \\ 1 & lb \end{bmatrix} = k(l\mathbf{u}).$$

OK.

(10) We have,

$$1 \cdot \mathbf{u} = \begin{bmatrix} 1 \cdot a & 1 \\ 1 & 1 \cdot b \end{bmatrix} = \mathbf{u}.$$

OK.

Hence, V is a vector space.

[**Note:** This vector space is just $\mathbb{R}^2$ disguised. The ones in the matrix play no role in anything here. We could then just as easily forget about then and identify:

$$\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} \in V \quad \longleftrightarrow \quad (a, b) \in \mathbb{R}^2.$$

Notice that the addition and scalar multiplication on V and on $\mathbb{R}^2$ are *"compatible"*, in other words,

$$\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} + \begin{bmatrix} c & 1 \\ 1 & d \end{bmatrix} = \begin{bmatrix} a+b & 1 \\ 1 & b+d \end{bmatrix} \quad \longleftrightarrow \quad (a, b) + (c, d) = (a+c, b+d),$$

and

$$k \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} = \begin{bmatrix} ka & 1 \\ 1 & kb \end{bmatrix} \quad \longleftrightarrow \quad k(a, b) = (ka, kb).]$$

Section 5.2

2. (b) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$ be two matrices with $a+b+c+d=0$ and $a'+b'+c'+d'=0$. Then,

$$A + B = \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix}.$$

Then, $(a+a')+(b+b')+(c+c')+(d+d') = (a+b+c+d)+(a'+b'+c'+d') = 0+0 = 0$. Hence, the subset is closed under addition.

Also, if $k \in \mathbb{R}$, we have that

$$kA = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix},$$

and $ka+kb+kc+kd = k(a+b+c+d) = k \cdot 0 = 0$. So, the subset is also closed under scalar multiplication. Therefore, it is a subspace of $M_{2\,2}$.

(b) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $\det A = \det B = 0$, but $A + B = \mathrm{Id}_2$, and hence $\det(A+B) = 1 \neq 0$. So, the subset is not closed under addition, and hence, it cannot be a subspace.

Remarks: Notice that to prove that a subset is *not* a subspace, it is usually better to give an example where either addition or scalar multiplication fails, as it was done above. On the other hand, to prove that a subset *is* a subspace, you *cannot* just give examples, and have to deal with the general case, as was done in part (a).

Note that if a subset fails to be only one between "closed under addition" or "closed under scalar multiplication" is enough to show it is not a subspace, and you don't have to look at the other at all, unless you are explicitly asked to check both. So, in this case, we did not check that if it is closed under scalar multiplication. [But, notice that it is, since $\det(kA) = k^2 \det A = k^2 \cdot 0 = 0$.]

4. (a) Let $f(x) = -x^2$. Then $f(x) \leq 0$ for all x. But $(-1)f(x) = x^2$ is not less than equal to zero for all x, for instance $(-1)f(1) = -(-1) = 1 > 0$. So, the subset is not closed under scalar multiplication, and hence not a subspace.

[You don't have to do this, but notice it is closed under addition, since if $f(x), g(x) \leq 0$ for all x, then $(f+g)(x) = f(x) + g(x)$ is a sum of two non-positive numbers for all x, since both $f(x)$ and $g(x)$ are.]

(b) Let $f(x)$ and $g(x)$ such that $f(0) = g(0) = 0$. Then, the function $(f+g)(x)$ is such that $(f+g)(0) = f(0) + g(0) = 0 + 0 = 0$. So, the subset is closed under addition.

Now, for all $k \in \mathbb{R}$, we have that the function $(kf)(x)$ is such that $(kf)(0) = k \cdot f(0) = k \cdot 0 = 0$. Thus, the subset is also closed under scalar multiplication, and hence it's a subspace.

(c) Let $f(x)$ be the function constant equals to 2. So, $f(0) = 2$, and hence it is in the given subset. Then $(2f)(0) = 2 \cdot f(0) = 4$, and hence it is not in the subset. So, the subset is not closed under scalar multiplication, and hence not a subspace.

[You don't have to do this, but notice it is also not closed under addition, since if $(f+f)(x) = f(x) + f(x) \equiv 4$ for all x, and in particular $(f+f)(0) = 4 \neq 2$.]

7. To save time, solve all systems simultaneously!

$$\left[\begin{array}{cc|c|c|c|c} 1 & 0 & 2 & 3 & 0 & 0 \\ 3 & -2 & 2 & 1 & 4 & 0 \\ -1 & 2 & 2 & 5 & 5 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c|c|c|c} 1 & 0 & 2 & 3 & 0 & 0 \\ 0 & -2 & -4 & -8 & 4 & 0 \\ 0 & 2 & 4 & 8 & 5 & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{cc|c|c|c|c} 1 & 0 & 2 & 3 & 0 & 0 \\ 0 & -2 & -4 & -8 & 4 & 0 \\ 0 & 0 & 0 & 0 & 9 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c|c|c|c} 1 & 0 & 2 & 3 & 0 & 0 \\ 0 & 1 & 2 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 & 9 & 0 \end{array}\right]$$

So, only (c) is not a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

[We were not asked here to find the scalars that give the linear combination, so we could have stopped after the second step of row operations above, since it already shows that (c) is the only one with no solutions. But, with the last step, we see that:

$$(2,2,2) = 2\mathbf{u} + 2\mathbf{v}, \quad (3,1,4) = 4\mathbf{u} + 3\mathbf{v}, \quad (0,0,0) = 0\mathbf{u} + 0\mathbf{v}.]$$

Section 5.3

6. (b) Since $\mathbf{v}_2 = -2\mathbf{v}_1$, we have that $\mathbf{v}_1$ and $\mathbf{v}_2$ are on the same line. But since $\mathbf{v}_3$ is not a multiple of $\mathbf{v}_1$ or $\mathbf{v}_2$, then $\mathbf{v}_3$ is not on the same line as $\mathbf{v}_1$ and $\mathbf{v}_2$.

(c) Since neither $\mathbf{v}_2$ nor $\mathbf{v}_3$ is a multiple of $\mathbf{v}_1$, neither $\mathbf{v}_2$ nor $\mathbf{v}_3$ is on the same line as $\mathbf{v}_1$. Also, since $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_3$, these two are not on the same line either.

[We were not asked this here, but notice that although no pair of vectors are on the same line, they are all on the same *plane*. In this case, this boils down to say that they are linearly dependent. But:

$$\begin{vmatrix} 2 & 4 & 2 \\ -1 & 2 & 7 \\ 4 & 3 & -6 \end{vmatrix} = 0,$$

and hence they are linearly dependent.]

8. (a) We have to find $k_1, k_2, k_3 \in \mathbb{R}$, not all zero, such that $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$. This gives us the linear system:

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 3 & 0 & 3 \\ 4 & -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solving the system [by row reduction, for instance], gives us that $(k_1, k_2, k_3) = (t, t, -t)$ for all $t \in \mathbb{R}$ are all the solutions. In particular, taking $t = 1$ [for instance], gives us a linear combination with not all coefficients equal to zero, namely:

$$\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}.$$

(b) With the equation above, this part is quite easy. Just solve for each one of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, to obtain:

$$\mathbf{v}_1 = \mathbf{v}_3 - \mathbf{v}_2,$$
$$\mathbf{v}_2 = \mathbf{v}_3 - \mathbf{v}_1,$$
$$\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2.$$

11. If a nonempty subset of S is linearly dependent, then either it has only a zero vector, or one of its vectors is a linear combination of the others [by Theorem 5.3.1]. If the former holds, then S itself has the zero vector. If the latter, one of the vectors in S is also a linear combination of other vectors in S [since all vectors in the subset are in S], and hence S is linearly dependent.

13. We can just use problem 12 above. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}, \ldots, \mathbf{v}_n\}$ were linearly independent, then every subset would also have to be by problem 12. But the subset $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is not, so $\{\mathbf{v}_1, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}, \ldots, \mathbf{v}_n\}$ has to be linearly dependent.

[There are many other ways to do this problem. Here is another one. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly dependent, then there are $k_1, \ldots, k_r \in \mathbb{R}$, not all zero, such that

$$k_1\mathbf{v}_1 + \cdots + k_r\mathbf{v}_r = \mathbf{0}.$$

Then, obviously,

$$k_1\mathbf{v}_1 + \cdots + k_r\mathbf{v}_r + 0\mathbf{v}_{r+1} + \cdots + 0\mathbf{v}_n = \mathbf{0},$$

and not all coefficients are zero [since not all k_i's are zero], and hence $\{\mathbf{v}_1, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}, \ldots, \mathbf{v}_n\}$ is linearly dependent.]

15. Suppose that

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}.$$

If $k_3 \neq 0$, then we would have that

$$\mathbf{v}_3 = -\frac{k_1}{k_3}\mathbf{v}_1 - \frac{k_2}{k_3}\mathbf{v}_2,$$

and we would have $\mathbf{v}_3 \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$, which cannot happen. So, k_3 must be zero.

Now, since $k_3 = 0$, the our first equation becomes

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 = \mathbf{0}.$$

But, since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent, we get that $k_1 = k_2 = 0$.

So, $k_1 = k_2 = k_3 = 0$, is the only possibility, and thus, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

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Section 5.4

4. (a) We know that $\dim P_2 = 3$. So, by Theorem 5.4.5, it suffices to check that these 3 vectors are either linearly independent or span P_2. [In general, one has to do both.] Of course, if one of these fails then, the other must fail too.

So, let's check if the polynomials are linearly independent. Let $k_1, k_2 k_3 \in \mathbb{R}$. Then, if

$$k_1(1-3x+2x^2)+k_2(1+x+4x^2)+k_3(1-7x)=0,$$

is the same as

$$(k_1+k_2+k_3)+(-3k_1+k_2-7k_3)x+(2k_1+4k_2)x^2=0.$$

Now, this only happens if the coefficients are zero, i.e., if

$$\begin{aligned} k_1 + \ \ k_2 + \ \ k_3 &= 0 \\ -3k_1 + \ \ k_2 - 7k_3 &= 0 \\ 2k_1 + 4k_2 \qquad &= 0 \end{aligned}$$

Note that we don't really need to solve this system, but only find if there is a non-trivial solution or not. So, we can find that simply by computing the determinant of the matrix of coefficients. But,

$$\begin{vmatrix} 1 & 1 & 1 \\ -3 & 1 & -7 \\ 2 & 4 & 0 \end{vmatrix} = 0.$$

So, there is a non-trivial solution, and hence the vectors are linearly dependent. Therefore, the polynomials do not make a basis of P_2.

(c) We take the same approach as the previous item. So, we check for linear dependency:

$$k_1(1+x+x^2)+k_2(x+x^2)+k_3(x^2)=0,$$

which is the same as

$$k_1+(k_1+k_2)x+(k_1+k_2+k_3)x^2=0.$$

So, we look at the system:

$$\begin{aligned} k_1 \qquad\quad &= 0 \\ k_1 + k_2 \qquad &= 0 \\ k_1 + k_2 + k_3 &= 0 \end{aligned}$$

But one can clearly see that this system has only the trivial solution [i.e., $k_1 = k_2 = k_3 = 0$]. [Or, one could also compute its determinant and find that it is not zero, in fact it's 1, and hence it has only the trivial solution.]

So, the polynomials are linearly independent. By Theorem 5.4.5, they form a basis.

8.(c) Since $\mathbf{w} = \mathbf{u}_2$, it is easy to see that $(\mathbf{w})_S = (0, 1)$ [since in general (a, b) corresponds to $a \cdot \mathbf{u}_1 + b \cdot \mathbf{u}_2$].

But one can also do it in the conventional way [which works in general]. We solve $\mathbf{w} = k_1\mathbf{u}_1 + k_2\mathbf{u}_2$ for k_1 and k_2. [After these are found, $(\mathbf{w})_S = (k_1, k_2)$.] So, we have $(1, 1) = (k_1, -k_1) + (k_2, k_2) = (k_1 + k_2, -k_1 + k_2)$, giving the system:

$$\begin{aligned} k_1 + k_2 &= 1 \\ -k_1 + k_2 &= 1 \end{aligned}$$

Solving it, we obtain $k_1 = 0$, $k_2 = 1$. Thus, $(\mathbf{w})_S = (0, 1)$.

10.(b) We want $k_1, k_2, k_3 \in \mathbb{R}$ such that $\mathbf{p} = k_1\mathbf{p}_1 + k_2\mathbf{p}_2 + k_3\mathbf{p}_3$ [and then $(\mathbf{p})_S = (k_1, k_2, k_3)$]. This, gives the system

$$\begin{aligned} k_1 + k_2 &= 2 \\ k_1 + k_3 &= -1 \\ k_2 + k_3 &= 1 \end{aligned}$$

The solution [using row reduction or any other method] is $k_1 = 0$, $k_2 = 2$, and $k_3 = -1$. Thus, $(\mathbf{p})_S = (0, 2, -1)$.

14. Solving the system [note that the second equation is a multiple of the first], we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4r - 3s + t \\ r \\ s \\ t \end{bmatrix} = r \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, since $S = \{(4, 1, 0, 0), (-3, 0, 1, 0), (1, 0, 0, 1)\}$ is linearly independent [since the solution is trivial only when $r = s = t = 0$], we have that S is basis [since it also clearly spans the set of solutions], and the dimension is 3.

16. Solving the system, one sees that the only solution is the trivial solution $(x_1, x_2, x_3) = (0, 0, 0)$. [One can just check that the determinant of the matrix of coefficients is non-zero, more precisely -8, and hence the homogeneous system only has the trivial solution.] Therefore, the space of solutions only has the zero vector $\mathbf{0}$. The dimension is then 0 [note, a space of dimension one is a *line*, while a single point is zero dimensional!], and it has no basis [note that $\{\mathbf{0}\}$ is not linearly independent, and hence cannot be a basis].

18. (a) We can take two approaches here. First we can just "solve the system" [of one equation and *three* variables, since we are in $\mathbb{R}^3$] $x - y = 0$. This gives us the solution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} s \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore, $S = \{(1, 1, 0), (0, 0, 1)\}$ is a basis.

The other way is to remember that a plane is two dimensional, so by Theorem 5.4.5, we just need to find two linearly independent vectors in $\mathbb{R}^3$ which are on the plane [i.e., such that $x - y = 0$]. These can be easily found by inspection, for instance, the set S above. [Note that we know that S is linearly independent, since no vector in it is not multiple of the other.] A couple of different possibilities would be $\{(2, 2, 1), (1, 1, 1)\}$, and $\{(1, 1, 1), (-1, -1, 1)\}$.

(c) We can use the same two ideas above, but in this case, the solution vector is already given:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2t \\ -t \\ 4t \end{bmatrix} = t \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}.$$

Thus $S = \{(2, -1, 4)\}$ is a basis.

The other way is to use Theorem 5.4.5 again, remembering that the line is one dimensional. So, we just need to find one non-zero [for linearly independence] vector of $\mathbb{R}^3$ which is on the line. For this we just choose a non-zero value for t. For instance, $t = 1$ gives the same S as above. [But any $t \neq 0$ would give also a basis, like $\{(-2, 1, -4)\}$, when $t = -1$, or $\{(1, -1/2, 2)\}$, when $t = 1/2$.]

19. By Theorem 5.4.5 again, it suffices to show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \{\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3\}$ is either linearly independent or that it spans the same set as $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Let's do the former. Suppose that

$$k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + k_3\mathbf{u}_3 = k_1\mathbf{v}_1 + k_2(\mathbf{v}_1 + \mathbf{v}_2) + k_3(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) = \mathbf{0}.$$

So, this gives us

$$(k_1 + k_2 + k_3)\mathbf{v}_1 + (k_2 + k_3)\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}.$$

Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis, it is linearly independent. Hence, the above equation implies that

$$\begin{aligned} k_1 + k_2 + k_3 &= 0 \\ k_2 + k_3 &= 0 \\ k_3 &= 0 \end{aligned}$$

But clearly this system only has the trivial solution $k_1 = k_2 = k_3 = 0$.

Therefore, $k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + k_3\mathbf{u}_3 = \mathbf{0}$ only when $k_1 = k_2 = k_3 = 0$. And thus, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly independent, and, as observed above, this shows that it is a basis.

Section 5.5

2. Remember that

$$[\mathbf{c}_1 \ \cdots \ \mathbf{c}_n] \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{c}_1 + \cdots + x_n\mathbf{c}_n.$$

So, we have:

(c)

$$-1 \cdot \begin{bmatrix} -3 \\ 5 \\ 2 \\ 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 6 \\ -4 \\ 3 \\ 8 \end{bmatrix} + 5 \cdot \begin{bmatrix} 2 \\ 0 \\ -1 \\ 3 \end{bmatrix}$$

(d)

$$3 \cdot \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 0 \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} - 5 \cdot \begin{bmatrix} 5 \\ -8 \end{bmatrix}$$

6.(c) The nullspace of A is the space of solutions on $A\mathbf{x} = \mathbf{b}$. Putting A in reduced row echelon form, we get:

$$\begin{bmatrix} 1 & 0 & 1 & -\frac{2}{7} \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, the solution is

$$\mathbf{x} = \begin{bmatrix} -t + \frac{2}{7}s \\ -t - \frac{4}{7}s \\ t \\ s \end{bmatrix} = t \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} \frac{2}{7} \\ -\frac{4}{7} \\ 0 \\ 1 \end{bmatrix}$$

So, the basis is $\{(-1, -1, 1, 0), (2/7, -4/7, 0, 1)\}$.

8.(c) The row space is generated by the non-zero rows of the echelon form of the matrix. [Here it does *not* matter if it is *reduced* echelon form or just plain echelon form.] So, a basis is $\{(1, 0, 1, -2/7), (0, 1, 1, 4/7)\}$.

10.(c) To do this we *transpose* the matrix A:

$$A^{\mathrm{T}} = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 1 & 3 \\ 5 & 3 & 2 \\ 2 & 0 & 2 \end{bmatrix}$$

Then, we put it in row echelon form:

$$A^{\mathrm{T}} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the first two columns of this matrix are the one with the leading ones, the first two columns of A^{T} generate the column space of A^{T}, and hence the row space of A. So, the basis is $\{(1, 4, 5, 2), (2, 1, 3, 0)\}$.

[It was not asked in the problem, but we can easily find how the thrid row of A is generated as a linear combination of the first two. For that, we again use the echelon form of the transpose. We can clearly see that

$$(1, -1, 0, 0) = 1 \cdot (1, 0, 0, 0) + (-1) \cdot (0, 1, 0, 0).$$

Thus, the same coefficients work for the rows of A:

$$(-1,3,2,2) = 1 \cdot (1,4,5,2) + (-1) \cdot (2,1,3,0).]$$

12.(c) We put the vectors in *columns* [since we want a subset of them for a basis]. Then, we get

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 & 0 & -7 \\ -1 & 3 & -5 & 4 & 18 \\ 5 & 1 & 9 & 2 & 2 \\ 2 & 0 & 4 & -3 & -8 \end{bmatrix}.$$

Putting it in reduced row echelon form, we get

$$R = \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 & \mathbf{c}_4 & \mathbf{c}_5 \end{bmatrix}.$$

Then, the $\mathbf{c}_1$, $\mathbf{c}_2$, and $\mathbf{c}_4$ are the columns of R with leading ones. So, they form a basis for the column space of R [which is *different* from the column space of A], and $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is a basis for the column space of A.

So, we need write $\mathbf{v}_3$ and $\mathbf{v}_5$ as linear combinations of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_4$. To do that, we use R again. It's easy to see that:

$$\mathbf{c}_3 = 2\mathbf{c}_1 - 1\mathbf{c}_2, \qquad \text{and} \qquad \mathbf{c}_5 = -\mathbf{c}_1 + 3\mathbf{c}_2 + 2\mathbf{c}_4.$$

[These are easy to find since R is in reduced echelon form. Plain echelon form is easy too, but it's a little more work.] Hence,

$$\mathbf{v}_3 = 2\mathbf{v}_1 - 1\mathbf{v}_2, \qquad \text{and} \qquad \mathbf{v}_5 = -\mathbf{v}_1 + 3\mathbf{v}_2 + 2\mathbf{v}_4.$$

Note: This means that $(\mathbf{v}_3)_S = (2,-1,0)$ and $(\mathbf{v}_5)_S = (-1,3,2)$.

Section 5.6

2. Remember that if E is the echelon form of A, then the rank of A is the number of columns [or rows] with leading ones in E, and the nullity is the number of columns [and it *cannot* be rows here!!] without leading ones in E, since this is the number of free variables [or parameters] in the general solution of the homogeneous system $A\mathbf{x} = \mathbf{0}$.

(b) A has [reduced] echelon form

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So, $\text{rank}(A) = 1$, and $\text{nulltiy}(A) = 2$. Then they add up to 3, which is the number of columns of A.

[If you want to see the nullspace explicitly, here it is: by the echelon form above, the system $A\mathbf{x} = \mathbf{0}$ reduces to

$$x_1 - \frac{1}{2}x_3 = 0.$$

Hence, $x_2 = s$, $x_3 = t$ and $x_1 = t/2$, i.e.,

$$(x_1, x_2, x_3) = (t/2, s, t) = t(1/2, 0, 1) + s(0, 1, 0).$$

Thus, $\text{nulltiy}(A) = 2$.]

(c) A has [reduced] echelon form

$$\begin{bmatrix} 1 & 0 & 1 & -\frac{2}{7} \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, $\text{rank}(A) = 2$, and $\text{nulltiy}(A) = 2$. Then they add up to 4, which is the number of columns of A.

[If you want to see the nullspace explicitly, here it is: by the echelon form above, the system $A\mathbf{x} = \mathbf{0}$ reduces to

$$\begin{aligned} x_1 + \qquad x_3 - \frac{2}{7}x_4 &= 0 \\ x_2 + x_3 + \frac{4}{7}x_4 &= 0 \end{aligned}$$

Hence, $x_3 = s$, $x_4 = t$, $x_2 = -s - 4t/7$, and $x_1 = -s + 2t/7$, i.e.,

$$(x_1, x_2, x_3, x_4) = (-s + 2t/7, -s - 4t/7, s, t) = t(2/7, -4/7, 0, 1) + s(-1, -1, 1, 0).$$

Thus, $\text{nulltiy}(A) = 2$.]

4. Remember that the rank is the dimension of both the columns space and row space. Also, the nullity is the difference between the number of columns and the rank. Since $\text{rank}(A) = \text{rank}(A^{\text{T}})$, then the nullity of A^{T} is the difference between the number of *rows* and the rank. So, in general, if A is a $m \times n$ matrix [i.e., m rows and n columns] of rank n [remember that $r \leq \min(m, n)$], then:

$$\begin{aligned} \text{dim(col. spc. of A)} = \text{dim(row spc. of A)} &= r, \\ \text{nulltiy}(A) &= n - r, \\ \text{nulltiy}(A^{\text{T}}) &= m - r. \end{aligned}$$

So,

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
size of A	3×3	3×3	3×3	5×9	9×5	4×4	6×2
rank of A	3	2	1	2	2	0	2
dim. of col/row spc.	3	2	1	2	2	0	2
nullity of A	0	1	2	7	3	4	0
nullity of A^{T}	0	1	2	3	7	4	4

8. Note that here we don't care about the rank of $[A \mid \mathbf{b}]$, since the system always has a solution. Remember also that the nullity is equal to the number of parameters [or free variables], which is equal to the number of columns minus the rank. Note also that we have already found that in exercise 4 [since nullity of A is the dimension of the nullspace by definition].

We have:

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
size of A	3×3	3×3	3×3	5×9	9×5	4×4	6×2
rank of A	3	2	1	2	2	0	2
nullity/# of param.	0	1	2	7	3	4	0

12. As usual, to find the rank we need to put the matrices in echelon form. We just have to be careful with divisions, since we cannot divide by 0. [So, if you divide a row by $(1 - t)$, for instance, you could be dividing by 0 in the case $t = 1$.] So, we avoid as much as possible divisions by elements involving t.

(a) We perform row operations *without dividing by expressions having t*:

$$\begin{bmatrix} 1 & 1 & t \\ 1 & t & 1 \\ t & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & t \\ 0 & t-1 & 1-t \\ 0 & 1-t & 1-t^2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & t \\ 0 & t-1 & -(t-1) \\ 0 & 0 & 2-t-t^2 \end{bmatrix}.$$

We now check the possibilities. First, observe that $2-t-t^2 = -(t-1)(t+2)$. [So, the last row is zero if, and only if, t is either 1 or -2.]

If $t = 1$, we get:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and so $\text{rank}(A) = 0$.

If $t = -2$, then we get:

$$\begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

and so $\text{rank}(A) = 2$.

If $t \neq 1, -2$, then $(t-1)$ and $2-t-t^2$ are both non-zero, and hence we can divide the second row by $(t-1)$ and the third by $2-t-t^2$, giving

$$\begin{bmatrix} 1 & 1 & t \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix},$$

and so $\text{rank}(A) = 3$.

In summary:

$$\text{rank}(A) = \begin{cases} 1, & \text{if } t = 1; \\ 2, & \text{if } t = -2; \\ 3, & \text{if } t \neq 1, -2. \end{cases}$$

(b) We take the same approach as above:

$$\begin{bmatrix} t & 3 & -1 \\ 3 & 6 & -2 \\ -1 & -3 & t \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -t \\ 3 & 6 & -2 \\ t & 3 & -1 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 3 & -t \\ 0 & -3 & -2+3t \\ 0 & 3-3t & -1+t^2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -t \\ 0 & 1 & \frac{2}{3}-t \\ 0 & 0 & -3+5t-3t^2 \end{bmatrix}$$

So, we can see that the rank is at least two. Now, if $-3+5t-3t^2=0$, then the rank is 2, but it is 3 otherwise. But, $-3+5t-3t^2=0$ only for $t=1,3/2$.

So,

$$\text{rank}(A) = \begin{cases} 2, & \text{if } t = 1, 3/2; \\ 3, & \text{if } t \neq 1, 3/2. \end{cases}$$

13. Since the first and last rows are linearly independent [since one in not a multiple of the other] not matter what values r and s have [since they don't involve r and s], we have that the rank is at least 2. Thus, the rank can never be 1.

For the rank to be 2, we need that the two middle rows have to be a linear combination of the first two. But, if

$$k_1(1,0,0) + k_2(0,0,3) = (0,a,b),$$

then,

$$(k_1, 0, 3k_2) = (0, a, b),$$

i.e., $k_1 = 0$, $k_2 = b/3$ and $a = 0$. So, the second coordinate of the second and third rows would have to be zero. Thus, $r = 2$ and $s = 1$. Therefore, we have the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and so it has rank 2. [Note that $r = 2$ and $s = 1$ are the *only* values that make the rank 2.]

[Note that choosing either $r \neq 2$ or $s \neq 1$ gives rank 3. Of course, we can never have rank 4, since we only have 3 columns.]

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Section 6.1

3.(a) $\langle \mathbf{u}, \mathbf{v} \rangle = 3 \cdot (-1) - 2 \cdot 3 + 4 \cdot 1 + 8 \cdot 1 = 3.$

4.(a) $\langle \mathbf{p}, \mathbf{q} \rangle = -2 \cdot 4 + 1 \cdot 0 + 3 \cdot (-7) = -29.$

6. (a)

$$\begin{aligned}
\langle \mathbf{u}, \mathbf{v} \rangle &= \left(\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right) \\
&= \begin{bmatrix} 2u_1 + u_2 \\ -u_1 + 3u_2 \end{bmatrix} \cdot \begin{bmatrix} 2v_1 + v_2 \\ -v_1 + 3v_2 \end{bmatrix} \\
&= (2u_1 + u_2)(2v_1 + v_2) + (-u_1 + 3u_2)(-v_1 + 3v_2) \\
&= 5u_1v_1 - u_1v_2 - u_2v_1 + 10u_2v_2.
\end{aligned}$$

(b) $\langle \mathbf{u}, \mathbf{v} \rangle = 5 \cdot 0 \cdot 6 - 0 \cdot 2 - (-3) \cdot 6 + 10 \cdot (-3) \cdot 2 = -42.$

8.(b) We have:

$$\begin{aligned}
\langle \mathbf{u}, \mathbf{v} \rangle &= 4u_1v_1 + u_2v_1 + u_1v_2 + 4u_2v_2 \\
&= 4v_1u_1 + v_2u_1 + v_1u_2 + 4v_2u_2 \\
&= \langle \mathbf{v}, \mathbf{u} \rangle .
\end{aligned}$$

We also have:

$$\begin{aligned}
\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle &= \langle (u_1 + v_1, u_2 + v_2), (w_1, w_2) \rangle \\
&= 4(u_1 + v_1)w_1 + (u_2 + v_2)w_1 + (u_1 + v_1)w_2 + 4(u_2 + v_2)w_2 \\
&= (4u_1w_1 + u_2w_1 + u_1w_2 + 4u_2w_2) + (4v_1w_1 + v_2w_1 + v_1w_2 + 4v_2w_2) \\
&= \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle .
\end{aligned}$$

Also,

$$\begin{aligned}
\langle k\mathbf{u}, \mathbf{v} \rangle &= \langle (ku_1, ku_2), (v_1, v_2) \rangle \\
&= 4(ku_1)v_1 + (ku_2)v_1 + (ku_1)v_2 + 4(ku_2)v_2 \\
&= k(4v_1u_1 + v_2u_1 + v_1u_2 + 4v_2u_2) \\
&= k \langle \mathbf{v}, \mathbf{u} \rangle .
\end{aligned}$$

Finally,

$$\begin{aligned}\langle \mathbf{u}, \mathbf{u} \rangle &= 4u_1^2 + 2u_1u_2 + 4u_2^2 \\ &\geq 4u_1^2 - 2\,|u_1|\,|u_2| + 4u_2^2 \\ &\geq 4u_1^2 - 8\,|u_1|\,|u_2| + 4u_2^2 \\ &= (2\,|u_1| - 2\,|u_2|)^2 \\ &\geq 0.\end{aligned}$$

Now, if $\langle \mathbf{u}, \mathbf{u} \rangle = 4u_1^2 + 2u_1u_2 + 4u_2^2 = 0$, then we look at two cases. If $u_2 = 0$, then $4u_1^2 = 0$ and thus also $u_1 = 0$, i.e., $\mathbf{u} = (0, 0)$. If $u_2 \neq 0$, we divide by u_2, obtaining

$$4\frac{u_1}{u_2} + 2\frac{u_1}{u_2} + 1 = 0,$$

and the quadratic formula gives us that

$$\frac{u_1}{u_2} = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 4 \cdot 1}}{8},$$

but these are not *real* solutions. Hence, there is no vector $\mathbf{u} = (u_1, u_2)$ with $u_2 \neq 0$ such that $\langle \mathbf{u}, \mathbf{u} \rangle = 0$, and hence the only possibility is $\mathbf{u} = \mathbf{0}$.

Note: The inner product in this problem is the inner product associated to the matrix

$$\begin{bmatrix} -\sqrt{15}/2 & 0 \\ 1/2 & 2 \end{bmatrix},$$

but this is harder to see.

9. (a) The second part of Axiom 4 fails. For instance, if $\mathbf{v} = (0, 1, 0)$, then $\langle \mathbf{v}, \mathbf{v} \rangle = 0 \cdot 0 + 0 \cdot 0 = 0$, but of course $\mathbf{v} \neq \mathbf{0}$. [Note that the first part of Axiom 4 holds.]

(a) The first and second part of Axiom 4 fails. [One would be enough!] For instance, if $\mathbf{v} = (0, 1, 0)$, then $\langle \mathbf{v}, \mathbf{v} \rangle = 0 \cdot 0 - 1 \cdot 1 + 0 \cdot 0 = -1 < 0$. Also, if $\mathbf{w} = (1, 1, 0)$, then $\langle \mathbf{w}, \mathbf{w} \rangle = 1 \cdot 1 - 1 \cdot 1 + 0 \cdot 0 = 0$, but of course $\mathbf{w} \neq \mathbf{0}$.

10.(c) We have

$$\begin{aligned}\|\mathbf{w}\|^2 &= \langle \mathbf{w}, \mathbf{w} \rangle \\ &= \left(\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right) \\ &= \begin{bmatrix} 5 \\ 10 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 10 \end{bmatrix} \\ &= 125.\end{aligned}$$

Thus, $\|\mathbf{w}\| = \sqrt{125} = 5\sqrt{5}$.

11. We have

$$\begin{aligned}\|\mathbf{w}\|^2 &= \langle \mathbf{w}, \mathbf{w} \rangle \\ &= \left(\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right) \\ &= \begin{bmatrix} 5 \\ 10 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 10 \end{bmatrix} \\ &= 125.\end{aligned}$$

Thus, $\|\mathbf{w}\| = \sqrt{125} = 5\sqrt{5}$.

14. We have

$$\begin{aligned} d(\mathbf{p}, \mathbf{q}) &= \|\mathbf{p} - \mathbf{q}\| \\ &= \left\|1 - x - 4x^2\right\| \\ &= \sqrt{1^2 + (-1)^2 + (-4)^2} \\ &= \sqrt{18} = 3\sqrt{2}. \end{aligned}$$

16. (c) We have:

$$\begin{aligned} \langle \mathbf{u} - \mathbf{v} - 2\mathbf{w}, 4\mathbf{u} + \mathbf{v} \rangle &= 4\langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle - 4\langle \mathbf{v}, \mathbf{u} \rangle - \langle \mathbf{v}, \mathbf{v} \rangle - 8\langle \mathbf{w}, \mathbf{u} \rangle - 2\langle \mathbf{w}, \mathbf{v} \rangle \\ &= 4\|\mathbf{u}\|^2 - 3\langle \mathbf{u}, \mathbf{v} \rangle - \|\mathbf{v}\|^2 - 8\langle \mathbf{u}, \mathbf{w} \rangle - 2\langle \mathbf{v}, \mathbf{w} \rangle \\ &= 4 - 6 - 4 - 40 - (-6) = -40. \end{aligned}$$

(d) We have:

$$\begin{aligned}\|\mathbf{u}-2\mathbf{v}+4\mathbf{w}\|^2 &= \langle \mathbf{u}-2\mathbf{v}+4\mathbf{w}, \mathbf{u}-2\mathbf{v}+4\mathbf{w}\rangle \\ &= \langle \mathbf{u},\mathbf{u}\rangle + 4\langle \mathbf{v},\mathbf{v}\rangle + 16\langle \mathbf{w},\mathbf{w}\rangle - 4\langle \mathbf{u},\mathbf{v}\rangle + 8\langle \mathbf{u},\mathbf{w}\rangle - 16\langle \mathbf{v},\mathbf{w}\rangle \\ &= \|\mathbf{u}\|^2 - 4\|\mathbf{v}\|^2 + 16\|\mathbf{w}\|^2 - 4\langle \mathbf{u},\mathbf{v}\rangle + 8\langle \mathbf{u},\mathbf{w}\rangle - 16\langle \mathbf{v},\mathbf{w}\rangle \\ &= 1 - 16 + 784 - 8 + 40 + 48 = 849\end{aligned}$$

So, $\|\mathbf{u}-2\mathbf{v}+4\mathbf{w}\| = \sqrt{849}$.

Section 6.2

2. If $\mathbf{u}$ and $\mathbf{w}$ are orthogonal, we must have

$$\langle \mathbf{u},\mathbf{w}\rangle = 2\cdot 1 + k\cdot 2 + 6\cdot 3 = 0,$$

and hence $k = -10$. If $\mathbf{v}$ and $\mathbf{w}$ are orthogonal, we must have

$$\langle \mathbf{v},\mathbf{w}\rangle = l\cdot 1 + 5\cdot 2 + 3\cdot 3 = 0,$$

and hence $k = -19$. But then,

$$\langle \mathbf{v},\mathbf{w}\rangle = 2\cdot(-19) + (-10)\cdot 5 + 3\cdot 6 = -70 \neq 0,$$

and so $\mathbf{v}$ and $\mathbf{w}$ are not orthogonal.

Therefore, there are no values of k and l that makes all vectors orthogonal to each other. [If $k \neq -10$, then $\mathbf{u}$ and $\mathbf{w}$ are not orthogonal. If $l \neq -19$, then $\mathbf{v}$ and $\mathbf{w}$ are not orthogonal. But if $k = -10$ and $l = -19$, then $\mathbf{v}$ and $\mathbf{w}$ are not orthogonal.]

6.(a) Let θ be the angle between $\mathbf{p}$ and $\mathbf{q}$. We have:

$$\langle \mathbf{p},\mathbf{q}\rangle = (-1)\cdot 2 + 5\cdot 4 + 2\cdot(-9) = 0.$$

Thus,

$$\cos(\theta) = \frac{\langle \mathbf{p},\mathbf{q}\rangle}{\|\mathbf{p}\|\,\|\mathbf{q}\|} = 0.$$

[We don't need to even compute the $\|\mathbf{p}\|$ and $\|\mathbf{q}\|$ in this case!] Hence, $\mathbf{p}$ and $\mathbf{q}$ are in fact orthogonal.

8.(a) Let θ be the angle between A and B. We have:

$$\begin{aligned}\langle A,B\rangle &= 2\cdot 3+6\cdot 2+1\cdot 1+(-3)\cdot 0=19,\\ \|A\|^2 &= \langle A,A\rangle = 2^2+6^2+1^2+(-3)^2=50,\\ \|B\|^2 &= \langle B,B\rangle = 3^2+2^2+1^2+0^2=14.\end{aligned}$$

So,

$$\cos(\theta)=\frac{\langle A,B\rangle}{\|A\|\,\|B\|}=\frac{19}{\sqrt{50}\sqrt{14}}=\frac{19\sqrt{7}}{70}.$$

11. We want a vector $\mathbf{x}$ such that $\mathbf{u}\cdot\mathbf{x}=\mathbf{v}\cdot\mathbf{x}=\mathbf{w}\cdot\mathbf{x}=0$ and $\|\mathbf{x}\|=1$. But then, we want have that

$$\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\\ \mathbf{w}\end{bmatrix}\mathbf{x}=\begin{bmatrix}\mathbf{u}\cdot\mathbf{x}\\ \mathbf{v}\cdot\mathbf{x}\\ \mathbf{w}\cdot\mathbf{x}\end{bmatrix}=\mathbf{0}.$$

So, we need to solve this system. We have:

$$A\overset{\text{def}}{=}\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\\ \mathbf{w}\end{bmatrix}=\begin{bmatrix}2&1&-4&0\\ -1&-1&2&2\\ 3&2&5&4\end{bmatrix}\sim\begin{bmatrix}1&0&0&\frac{34}{11}\\ 0&1&0&-4\\ 0&0&1&\frac{6}{11}\end{bmatrix}.$$

[Note that what we have just done in here was to find the orthogonal compliment of $W\overset{\text{def}}{=}\operatorname{span}\{\mathbf{u},\mathbf{v},\mathbf{w}\}$. In fact, in view of Problem 21 below, one could have taken this approach from the start, since then we know that if $\mathbf{x}$ is orthogonal to $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, then is must be orthogonal to the subspace W. Hence, we have that $\mathbf{x}\in W^{\perp}$.]

Hence, the solution of $A\mathbf{x}=\mathbf{0}$ is $(-34t/11,4t,-6t/11,t)$, with $t\in\mathbb{R}$. So, $\mathbf{x}$ is orthogonal to $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ if, and only if, $\mathbf{x}$ is a multiple of $(-34,4,-6,11)$. [We can multiply the vector by 11 to clear denominators here. Also, note that this also means that $W^{\perp}=\operatorname{span}\{(-34,4,-6,11)\}$.]

Now, if $\|\mathbf{x}\|=\|t(-34,4,-6,11)\|=1$ [since we want *unit vectors*], we have

$$|t|\sqrt{(-34)^2+4^2(-6)^2+11^2}=|t|\,57=1.$$

So, the unit vectors orthogonal to $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are $\pm 1/57(-34,4,-6,11)$. [Note that these are the *only* unit vectors which are orthogonal to all three.]

15. (a) To find the vectors that span the plane, we just solve its equation: the system [with one equation and three variables] $x-2y-3z=0$ has solution $s(2,1,0)+t(3,0,1)$, where $s, t \in \mathbb{R}$. [This is a *parametrization of the plane*, for those who have taken, or are taking, Math 241.] Hence, this plane is $\operatorname{span}\{(2,1,0),(3,0,1)\}$. So, to find $W^{\perp}$ we find the nullspace of

$$\begin{bmatrix} 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \end{bmatrix}.$$

So, $W^{\perp} = \operatorname{span}\{(-1,2,3)\}$.

(b) We have that $W = \operatorname{span}\{(2,-5,4)\}$. So, find the nullspace of the matrix $\begin{bmatrix} 2 & -5 & 4 \end{bmatrix}$, we get $W^{\perp} = \operatorname{span}\{(-2,0,1),(5,2,0)\}$.

(c) So, W is the solution space of the system

$$\begin{aligned} x + y + z &= 0 \\ x - y + z &= 0 \end{aligned}$$

So, $W = \operatorname{span}\{(1,0,-1)\}$. We get $W^{\perp} = \operatorname{span}\{(1,0,1),(0,1,0)\}$.

21. Since $\mathbf{w}$ is orthogonal to all $\mathbf{u}_i$, we have:

$$\langle \mathbf{w}, \mathbf{u}_1 \rangle = \langle \mathbf{w}, \mathbf{u}_2 \rangle = \cdots = \langle \mathbf{w}, \mathbf{u}_r \rangle = 0.$$

If $\mathbf{v} \in \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$, then there are $k_1, \ldots, k_r \in \mathbb{R}$ such that

$$\mathbf{v} = k_1\mathbf{u}_1 + \cdots + k_r\mathbf{u}_r.$$

But then,

$$\begin{aligned} \langle \mathbf{w}, \mathbf{v} \rangle &= \langle \mathbf{w}, k_1\mathbf{u}_1 + \cdots + k_r\mathbf{u}_r \rangle \\ &= \langle \mathbf{w}, k_1\mathbf{u}_1 \rangle + \cdots + \langle \mathbf{w}, k_r\mathbf{u}_r \rangle \\ &= k_1 \langle \mathbf{w}, \mathbf{u}_1 \rangle + \cdots + k_r \langle \mathbf{w}, \mathbf{u}_r \rangle \\ &= 0 + \cdots + 0 = 0. \end{aligned}$$

Thus, $\mathbf{w}$ is orthogonal to every vector in $\operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$.

Section 6.3

2. Only (b).

4. (b) and (d).

6. (a) Orthonormal.

(b) Orthogonal, but not orthonormal, since, for example,

$$\left\| \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\| = \sqrt{0^2 + 0^2 + 1^2 + 1^2} = \sqrt{2} \neq 1.$$

8. We have

$$\langle (1,0), (0,1) \rangle = 4 \cdot 1 \cdot 0 + 0 \cdot 1 = 0.$$

So, the vectors are orthogonal. To make them orthonormal, we need to divide each by its length *with respect to the given inner product* [not the usual Euclidean one]. We have:

$$\|(1,0)\| = \sqrt{\langle (1,0), (1,0) \rangle} = \sqrt{4 \cdot 1 \cdot 1 + 0 \cdot 0} = 2$$

and

$$\|(0,1)\| = \sqrt{\langle (0,1), (0,1) \rangle} = \sqrt{4 \cdot 0 \cdot 0 + 1 \cdot 1} = 1.$$

So, the orthonormal set would be $\{(1/2, 0), (0,1)\}$.

14. Just remember that since S is an orthonormal basis, lengths and inner product can be computed from the *coordinate vectors* with respect to S as if we had the Euclidean inner product [i.e., the usual dot product].

(a) We have:

$$\begin{aligned}
\|\mathbf{u}\| &= \sqrt{(-1)^2 + 2^2 + 1^2 + 3^2} = \sqrt{15}, \\
\|\mathbf{v} - \mathbf{w}\| &= \sqrt{(0 - (-2))^2 + (-3 - (-4))^2 + (1 - 3)^2 + (5 - 1)^2} = 5, \\
\|\mathbf{v} + \mathbf{w}\| &= \sqrt{(0 + (-2))^2 + (-3 + (-4))^2 + (1 + 3)^2 + (5 + 1)^2} = \sqrt{105}, \\
\langle \mathbf{v}, \mathbf{w} \rangle &= 0 \cdot (-2) + (-3) \cdot (-4) + 1 \cdot 3 + 5 \cdot 1 = 20.
\end{aligned}$$

(b) We have:

$$\begin{aligned}
\|\mathbf{u}\| &= \sqrt{0^2 + 0^2 + (-1)^2 + (-1)^2} = \sqrt{2}, \\
\|\mathbf{v} - \mathbf{w}\| &= \sqrt{(5 - 3)^2 + (5 - 0)^2 + (-2 - (-3))^2 + (-2 - 0)^2} = \sqrt{34}, \\
\|\mathbf{v} + \mathbf{w}\| &= \sqrt{(5 + 3)^2 + (5 + 0)^2 + (-2 + (-3))^2 + (-2 + 0)^2} = \sqrt{118}, \\
\langle \mathbf{v}, \mathbf{w} \rangle &= 5 \cdot 3 + 5 \cdot 0 + (-2) \cdot (-3) + (-2) \cdot 0 = 21.
\end{aligned}$$

Section 6.5

2. **Note:** We *cannot* use Theorem 6.3.1 to compute the coordinates of a vector with respect to a basis S [by computing inned products] if S is not orthonormal!

(a) We want to find $k_1, k_2, k_3 \in \mathbb{R}$ such that

$$\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3.$$

This gives us the system:

$$\begin{aligned} k_1 + 2k_2 + 3k_3 &= 2 \\ 2k_2 + 3k_3 &= -1 \\ 3k_3 &= 3 \end{aligned}$$

Solving, we obtain $k_3 = 1$, $k_2 = -2$, and $k_1 = 3$. Thus,

$$(\mathbf{v})_S = (k_1, k_2, k_3) = (3, -2, 1).$$

[**Note:** It is not worth computing the transition matrix [from the standard basis to S] in this example, since we just want to find the coordinates of *one* vector with respect to S. Finding the transition matrix would require solving three [simple] systems [instead of only one], and then multiply the vector $(2, -1, 3)$ by this matrix.]

(b) We want to find $k_1, k_2, k_3 \in \mathbb{R}$ such that

$$\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3.$$

This gives us the system:

$$\begin{aligned} k_1 - 4k_2 + 7k_3 &= 5 \\ 2k_1 + 5k_2 - 8k_3 &= -12 \\ 3k_1 + 6k_2 + 9k_3 &= 3 \end{aligned}$$

Solving [as usual, like using row reduction], we obtain $k_1 = -2$, $k_2 = 0$, and $k_3 = 1$. Thus,

$$(\mathbf{v})_S = (k_1, k_2, k_3) = (-2, 0, 1).$$

4. See the note from Problem 2 above!

We want $k_1, k_2, k_3, k_4 \in \mathbb{R}$ such that

$$\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix} = k_1 \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + k_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + k_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + k_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

This gives us the system

$$\begin{aligned} -k_1 + k_2 \qquad\qquad &= 2 \\ k_1 + k_2 \qquad\qquad &= 0 \\ k_3 \; &= -1 \\ k_4 &= 3 \end{aligned}$$

So, $k_1 = -1$, $k_2 = 1$, $k_3 = -1$, $k_4 = 3$, i.e.,

$$\left(\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}\right)_S = (-1, 1, -1, 3).$$

6. (a) Note that B is the standard basis of $\mathbb{R}^2$, which makes things a little easier, since it gives us that $((a,b))_B = (a,b)$. To find the transition matrix from B' to B, we find $(\mathbf{v}_1)_B$ and $(\mathbf{v}_2)_B$ and put those as columns. By our previous remark, this is quite easy, then yielding:

$$P \stackrel{\text{def}}{=} \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}.$$

(b) For this one, we can either use the analogous procedure from (a) [i.e., find $(\mathbf{u}_1)_{B'}$ and $(\mathbf{u}_2)_{B'}$ and put those as columns] or we can simply use Theorem 6.5.1, which tells us that the matrix we are looking for is P^{-1} [with P from part (a)]. In either computation we get

$$P^{-1} = \frac{1}{11} \begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{2}{11} \end{bmatrix}.$$

(c) As observed in (a), we have that

$$[\mathbf{w}]_B = \left[\begin{bmatrix} 3 \\ -5 \end{bmatrix}\right]_B = \begin{bmatrix} 3 \\ -5 \end{bmatrix}.$$

So, we now have:

$$[\mathbf{w}]_{B'} = \begin{bmatrix} \frac{4}{11} & \frac{3}{11} \\ -\frac{1}{11} & \frac{2}{11} \end{bmatrix} \cdot [\mathbf{w}]_B = \begin{bmatrix} -\frac{3}{11} \\ -\frac{13}{11} \end{bmatrix}$$

(d) We want $k_1, k_2 \in \mathbb{R}$ such that $\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2$. This gives us the system:

$$\begin{aligned} 2k_1 - 3k_2 &= 3 \\ k_1 + 4k_2 &= -5 \end{aligned}$$

We solve this in the usual manner, for instance, the augmented matrix has reduced echelon form:

$$\left[\begin{array}{cc|c} 1 & 0 & -\frac{3}{11} \\ 0 & 1 & -\frac{13}{11} \end{array}\right]$$

giving $k_1 = -3/11$ and $k_2 = -13/11$. Hence, $(\mathbf{w})_{B'} = (-3/11, -13/11)$ as in (c).

11. Note that

$$V = \{k_1 \sin(x) + k_2 \cos(x) \ : \ k_1, k_2 \in \mathbb{R}\}.$$

Also note that $\{\sin(x), \cos(x)\}$ is a linearly independent set, for if $k_1 \sin(x) + k_2 \cos(x) = 0$, then plugging in $x = 0$ we get that $k_2 = 0$, and plugging in $x = \pi/2$, we get that $k_1 = 0$. So, there is no non-trivial [i.e., with not all zero scalars] linear combination of $\sin(x)$ and $\cos(x)$ that results in 0, which means that the set is linearly independent.

(a) Since we have two vectors exactly [which is the dimension of V, since $\{\sin(x), \cos(x)\}$ is a basis], it suffices to show that $\{\mathbf{g}_1, \mathbf{g}_2\}$ either is linearly independent or that it spans V. Let's do the latter: let $\mathbf{f} \overset{\text{def}}{=} k_1 \sin(x) + k_2 \cos(x)$ be an [arbitrary] element of V. We need to find $l_1, l_2 \in \mathbb{R}$ such that $l_1\mathbf{g}_1 + l_2\mathbf{g}_2 = \mathbf{f}$. This gives us:

$$2l_1 \sin(x) + (l_1 + 3l_2)\cos(x) = k_1 \sin(x) + k_2 \cos(x).$$

So, we have the system [in which we wan to find l_1 and l_2 given k_1 and k_2]:

$$\begin{aligned} 2l_1 \qquad &= k_1 \\ l_1 + 3l_2 &= k_2 \end{aligned}$$

So, we get $l_1 = k_1/2$ and $l_2 = k_2/3 - k_1/6$. Therefore, the system always has a solution, and thus $\{\mathbf{g}_1, \mathbf{g}_2\}$ spans V.

(b) It is very easy to see that $\mathbf{g}_1 = 2\mathbf{f}_1 + \mathbf{f}_2$ and $\mathbf{g}_2 = 3\mathbf{f}_2$. So, the transition matrix [from B' to B] is:

$$\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}.$$

(c) As before, we can just invert the matrix from (b). But note that our work in (a) makes it easy to find $(\mathbf{f}_1)_{B'}$ and $(\mathbf{f}_2)_{B'}$. We just use $k_1 = 1$ and $k_2 = 0$ for the former and $k_1 = 0$ and $k_2 = 1$ for the latter [and find the corresponding l_1 and l_2]. So, we have $(\mathbf{f}_1)_{B'} = (1/2, -1/6)$ and $(\mathbf{f}_2)_{B'} = (0, 1/3)$, giving the transition matrix

$$\begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{bmatrix}$$

(d) It's easy to see that $[\mathbf{h}]_B = (2, -5)$ [we just look at the coefficients of $\sin(x)$ and $\cos(x)$]. Now, we have

$$[\mathbf{h}]_{B'} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{bmatrix} \cdot [\mathbf{h}]_B = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

(e) We can again use (a) [since we already solved the general system], with $k_1 = 2$ and $k_2 = -5$. This gives us $l_1 = 2/2 = 1$ and $l_2 = -5/3 - 2/6 = -2$. So, $(\mathbf{h})_{B'} = (1, -2)$ as in (d).

Section 7.1

7. (a) $\lambda^4 + \lambda^3 - 3\lambda^2 - \lambda + 2 = (\lambda + 1) \cdot (\lambda + 2) \cdot (\lambda - 1)^2$.

(b) $\lambda^4 - 8\lambda^3 + 19\lambda^2 - 24\lambda + 48 = (\lambda - 4)^2 \cdot (\lambda^2 + 3)$.

9. (a) We have:

Eigenvalue (λ)	Basis of Eigenspace
-1	$\{(2, -1, -1, 0)\}$
-2	$\{(1, 0, -1, 0)\}$
1	$\{(2, 3, 1, 0), (0, 0, 0, 1)\}$

(b) Note that here we only consider *real* eigenvalues, so the only eigenvalue in this case is $\lambda = 4$, and the basis of the eigenspace is $\{(3, 2, 0, 0)\}$.

20. By assumption, we have that $A\mathbf{x} = \lambda\mathbf{x}$. Multiplying by A^{-1} we get $\mathbf{x} = A^{-1}(\lambda\mathbf{x}) = \lambda(A^{-1}\mathbf{x})$. Since A is invertible, by Theorem 7.1.4, we have that $\lambda \neq 0$. Hence, we can divide the equation by λ, getting $\lambda^{-1}\mathbf{x} = A^{-1}\mathbf{x}$, and hence $\mathbf{x}$ is an eigenvector of A^{-1} corresponding to the eigenvalue λ^{-1}.

Section 7.2

2. (a) We have:

$$\det(\lambda I - A) = \lambda^3 - 11\lambda^2 + 39\lambda - 45 = (\lambda - 5)(\lambda - 3)^2.$$

Hence, the eigenvalues are $\lambda = 5$ and $\lambda = 3$.

(b) We have for $\lambda = 3$:

$$3I - A = \begin{bmatrix} -1 & 0 & -1 \\ -2 & 0 & -2 \\ -1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So, $3I - A$ has rank 1.

For $\lambda = 5$:

$$5I - A = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 2 & -2 \\ -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

So, $5I - A$ has rank 2.

(c) We have that eigenspace corresponding to $\lambda = 3$ is the nullspace of $(3I - A)$. So, the dimension of this eigenspace is the nullity of $(3I - A)$. The nullity is equal to the number of columns minus the rank (Theorem 5.6.3), and hence the it is 2. Thus, the dimension of the eigenspace corresponding to 3 is 2.

In the same way, we can conclude that the dimension of the eigenspace corresponding to $\lambda = 5$ is 1.

So, putting together the bases for the eigenspaces we get 3 linearly independent vectors [by Theorem 7.2.2, or, more precisely, by the remark that follows its proof on pg. 374]. Thus, by Theorem 7.2.1, we have that A is diagonalizable.

[**Note:** We also know its diagonal form: has the eigenvalues in the diagonal, and you repeat the same eigenvalue the as many times as the dimension of its corresponding eigenspace. So, A can be diagonalized to

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

where these different possibilities come from different orderings of the basis made of eigenvectors.]

6. We have:

$$\det(\lambda I - A) = \lambda^3 - \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda^2 + \lambda + 1)$$

So, there is only one real eigenvalue, which is $\lambda = 2$. By Theorem 7.2.4, we have that the dimension of its eigenspace is 1. [Well, in fact the theorem gives us "at most 1", but we know that there is always a [non-zero] eigenvector for all eigenvalues, so we have that is is 1.] Hence, there are no three linearly independent eigenvector for A, and hence A is *not* diagonalizable.

[**Note:** We are not [and will not be] dealing with complex numbers in this course, but for those who are familiar with them, note that the characteristic equation has three distinct *complex* solutions: 2, $(-1 + \sqrt{3}\,\mathrm{i})/2$, and $(-1 - \sqrt{3}\,\mathrm{i})/2$. So, if we were to allow *complex* entries in vector and complex scalars, then this matrix *would* be diagonalizable.

Note that you do not need to worry about worry about complex numbers at all in this course. This was just for your information.]

Section 7.3

1. The book has the solutions. Just observe that in this case, since the matrices are invertible, we know that they are diagonalizable from the start. So, we know that the geometric multiplicity of an eigenvalue must be equal to its algebraic multiplicity.

5. We have

$$\det(\lambda I - A) = \lambda^3 + 28\lambda^2 - 1175\lambda - 3750 = (\lambda - 25)(\lambda + 3)(\lambda + 50).$$

So, we have 3 eigenvalues: $\lambda = 25, -3, -50$. With the usual procedure [i.e., finding nullspaces for $(\lambda I - A)$ for each λ], we find the eigenspaces. They are:

Eigenvalue (λ)	Basis of Eigenspace
25	$\{(4, 0, -3)\}$
-3	$\{(0, 1, 0)\}$
-50	$\{(3, 0, 4)\}$

So, we just need to make these basis vector have length 1 [since by Theorem 7.3.2, they are already orthogonal]. Dividing by their lengths, we get:

$$\{(4/5, 0, -3/5), (0, 1, 0), (3/5, 0, 4/5)\}$$

is an orthonormal basis of $\mathbb{R}^3$ made of eigenvalues of A. Then, the transition matrix diagonalizes A. This is given by [putting the vectors of this basis as columns]:

$$P \overset{\text{def}}{=} \begin{bmatrix} \frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ -\frac{3}{5} & 0 & \frac{4}{5} \end{bmatrix}.$$

We have then,

$$P^{\mathrm{T}} \cdot A \cdot P = \begin{bmatrix} 25 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -50 \end{bmatrix}.$$

[Careful with the order of the elements of the diagonal!]

[**Note:** The first column of the answer in the back of the book is -1 times the first column above. Both are correct, since multiplying by -1 keeps the length of the vector equal to 1 and does not change orthogonality.]

7. We have:

$$\det(\lambda I - A) = \lambda^3 - 6\lambda^2 + 9\lambda = \lambda(\lambda - 3)^2.$$

Computing the eigenspaces, we get:

Eigenvalue (λ)	Basis of Eigenspace
0	$\{(1, 1, 1)\}$
3	$\{(1, 0, -1), (0, 1, -1)\}$

Here we need to find an orthonormal basis for the eigenspace corresponding to $\lambda = 3$. To do that, we first replace the second vector in the basis by its *orthogonal component*.

[Remember: we have that the projection of $\mathbf{v}$ in the direction of $\mathbf{u}$ is given by

$$\operatorname{proj}_{\mathbf{u}} \mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|^2} \mathbf{u},$$

and the orthogonal component of $\mathbf{v}$ with respect to $\mathbf{u}$ is

$$\mathbf{v} - \operatorname{proj}_{\mathbf{u}} \mathbf{v}.]$$

We have

$$(0, 1, -1) - \operatorname{proj}_{(1,0,-1)}(0, 1, -1) = (0, 1, -1) - \frac{1}{2}(1, 0, -1) = \left(-\frac{1}{2}, 1, -\frac{1}{2}\right).$$

So, we have that $\{(1,0,-1),(1,-2,1)\}$ is an orthogonal basis of the eigenspace corresponding to $\lambda = 3$.

[I've multiplied the $(-1/2,1,-1/2)$ by -2 to simplify. This changes neither orthogonality, since it only changes length, nor being an eigenvector, since a non-zero scalar multiple of an eigenvector is also an eigenvector with the same eigenvalue. *It is not necessary to do it here*, though, since we will normalize in the next step. But it makes things a little easier.]

To make this basis orthogonal, we divide each vector by its length, obtaining

$$\left\{\left(\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right),\left(\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}}\right)\right\}.$$

Now, to make the basis of the eigenspace corresponding to $\lambda = 0$ orthonormal, we just need to divide the only vector in the basis by its length, giving $\{(1/\sqrt{3},1/\sqrt{3},1/\sqrt{3})\}$.

So, an orthonormal basis of eigenvalues of A is:

$$\left\{\left(\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right),\left(\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}}\right),\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)\right\}.$$

As before, the transition matrix gives us the orthogonal matrix that diagonalizes A, namely:

$$P \overset{\text{def}}{=} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}.$$

and

$$P^{\mathrm{T}} \cdot A \cdot P = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

[**Note:** Since the book used a different order for the basis [i.e., the columns of the matrix P are switched when compared to the one above], the elements on the main diagonal [i.e., the eigenvalues] are also in different order. But, of course, both the answer above and the on in the book are correct.]