1) [10 points] Put the following matrix in reduced row echelon form:

$$
\begin{bmatrix}
1 & 3 & -2 & 0 & 2 & 0 \\
2 & 6 & -5 & -2 & 4 & -3 \\
0 & 0 & 5 & 10 & 0 & 15 \\
2 & 6 & 0 & 8 & 4 & 18
\end{bmatrix}
$$

Solution. This is the coefficient matrix of the system in Example 4 on pg. 12 of the text. [Just follow the steps disregarding the last column.] The reduced echelon form is:

$$
\begin{bmatrix}
1 & 3 & 0 & 4 & 2 & 0 \\
0 & 0 & 1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$
2) [15 points] Let

\[ A = \begin{bmatrix} 4 & 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 & -3 \\ 2 & 1 & 2 & 3 & 0 \\ 1 & 2 & 0 & 2 & 6 \end{bmatrix} \, . \]

Compute \( \det(A) \).

Solution. We have:

\[
\det(A) = 1 \cdot \begin{vmatrix} 4 & 1 & 3 \\ 2 & 1 & 0 \\ 1 & 2 & 6 \end{vmatrix} \quad \text{[cofactors through the 2nd row]}
\]

\[
= (-1) \cdot \begin{vmatrix} 4 & 1 & 3 \\ 1 & -1 & -3 \\ 2 & 1 & 0 \end{vmatrix} + 2 \cdot \begin{vmatrix} 4 & 1 & 3 \\ 1 & 2 & 6 \\ 1 & -1 & -3 \end{vmatrix} \quad \text{[cofactors through the 3rd col]}
\]

\[
= (-1) \cdot (24 + 12 - (3 + 12)) + 2 \cdot 0 \quad \text{[Sarrus rule and col. mult. of another]}
\]

\[
= -21.
\]
3) [40 points] You should be able to answer the following questions quickly. You do not need to justify your answers.

(a) [4 points] Give the matrix that represents the rotation by $\pi/2$ about the $z$-axis, followed by a reflection about the $xz$-plane in $\mathbb{R}^3$.

Solution. We have:
\[
\begin{align*}
\mathbf{e}_1 & \rightarrow \mathbf{e}_2 \rightarrow -\mathbf{e}_2, \\
\mathbf{e}_2 & \rightarrow -\mathbf{e}_1 \rightarrow -\mathbf{e}_1, \\
\mathbf{e}_3 & \rightarrow \mathbf{e}_3 \rightarrow \mathbf{e}_3.
\end{align*}
\]
So, the matrix is \[
\begin{bmatrix}
-\mathbf{e}_2 & -\mathbf{e}_1 & \mathbf{e}_3
\end{bmatrix} = \begin{bmatrix}
0 & -1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

(b) [3 points] If $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, how many solutions can $A\mathbf{x} = \mathbf{b}$ possibly have?

Solution. It could have infinitely many or none at all.

(c) [3 points] If $A$ is an invertible $n$ by $n$ matrix, then what can we say about the reduced echelon form of $A$.

Solution. It is the identity matrix $I_n$.

(d) [3 points] Let $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be the linear transformation given by
\[
T(x_1, x_2, x_3, x_4) = (2x_1, -x_2, x_3, 3x_4).
\]
Give $[T^{-1}]$.

Solution.
\[
[T^{-1}] = [T]^{-1} = \begin{bmatrix}
2 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 3
\end{bmatrix}^{-1} = \begin{bmatrix}
1/2 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1/3
\end{bmatrix}.
\]
(e) [3 points] Let $T_A$ be the linear transformation associated to the $m$ by $n$ matrix $A$. If $T_A$ is onto, then what can we say about the rank of $A$? [If this rank is unrelated to whether or not $T_A$ is onto, just say so.]

Solution. The rank must be $m$.  

(f) [3 points] Is \( \{1 + x^2, 2 - x^3, 1 + x + x^2 + x^3\} \) a basis of $P_3$? Justify your answer in one short sentence.

Solution. No, since $\text{dim}(P_3) = 4$ and we only have 3 vectors in the set.  

(g) [3 points] If $B = \{e_1, e_2\}$ and $B' = \{(1,1), (2,1)\}$, find the transition matrix from $B$ to $B'$.

Solution. The matrix is:

\[
\begin{bmatrix}
1 & 2 \\
1 & 1 \\
\end{bmatrix}^{-1} = \frac{1}{-1} \cdot \begin{bmatrix}
1 & -2 \\
-1 & 1 \\
\end{bmatrix} = \begin{bmatrix}
-1 & 2 \\
1 & -1 \\
\end{bmatrix}.
\]

(h) [3 points] If $S = \{(1,0,1), (-2,1,1), (0,0,3)\}$ is a basis of $\mathbb{R}^3$, then the coordinates \( ((2,2,2))_S \) is given by the solution of what linear system? Give your answer in matrix form.

Solution.

\[
\begin{bmatrix}
1 & -2 & 0 \\
0 & 1 & 0 \\
1 & 1 & 3 \\
\end{bmatrix} x = \begin{bmatrix}
2 \\
2 \\
2 \\
\end{bmatrix}.
\]

(i) [3 points] What is the dimension of $M_{m \times n}$?

Solution. It is $m \cdot n$. 

(j) [3 points] Give the standard basis of $P_3$.

Solution. $\{1, x, x^2, x^3\}$. 

(k) [3 points] Let $S = \{v_1, v_2\}$ be an orthogonal, but not orthonormal, basis of a subspace $W$ of $V$, and $v \in V$, give the formula for $\text{proj}_W v$.

Solution. $\text{proj}_W v = \left(\frac{v_1}{||v_1||} \cdot v\right) \frac{v_1}{||v_1||} + \left(\frac{v_2}{||v_2||} \cdot v\right) \frac{v_2}{||v_1||}$. 

(l) [3 points] If $A$ is a 5 by 4 matrix of rank 3, give the nullities of $A$ and $A^T$.

Solution. Remember: rank plus nullity of $A$ is the number of columns, so the nullity of $A$ is 1. Also, rank plus nullity of $A^T$ is the number of columns of $A^T$, which is the number of rows of $A$. Hence, nullity of $A^T$ is 2. 

(m) [3 points] What condition on the size of the matrix $A$ guarantee that the system $Ax = 0$ has a non-trivial solution? [If there is no such condition, just say so.]

Solution. We need more variables than equations, so $A$ must have more columns than rows. [So, if $A$ is $m$ by $n$, then we need $n > m$.]
4) [15 points] Let

\[ A = \begin{bmatrix}
1 & 1 & 3 \\
0 & -2 & 0 \\
0 & 0 & -2
\end{bmatrix}. \]

(a) [3 points] Find the eigenvalues of \( A \). [You do not need to justify this one.]

**Solution.** Since the matrix is upper triangular, the eigenvalues are the elements in the main diagonal: 1 and \(-2\).

(b) [6 points] Find the eigenspaces associated to each eigenvalue.

**Solution.** For \( \lambda = 1 \) we have:

\[ 1 \cdot I_3 - A = \begin{bmatrix}
0 & -1 & -3 \\
0 & 3 & 0 \\
0 & 0 & 3
\end{bmatrix} \sim \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}. \]

So, we have that the eigenspace associated to 1 is \( \text{span}\{(1,0,0)\} \).

For \( \lambda = -2 \), we have:

\[ -2 \cdot I_3 - A = \begin{bmatrix}
-3 & -1 & -3 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \sim \begin{bmatrix}
1 & 1/3 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}. \]

So, we have that the eigenspace associated to \(-2\) is \( \text{span}\{(-1/3,1,0),(-1,0,1)\} \).

(c) [6 points] Is \( A \) diagonalizable? If so, give \( P \) such that \( P^{-1}AP \) is diagonal and the resulting diagonal form. [You do not need to justify in this case.] If not, explain why not.

**Solution.** Yes, since the dimensions of the eigenspaces add up to the number of rows. Then,

\[ P = \begin{bmatrix}
1 & -1/3 & -1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix}
1 & 0 & 0 \\
0 & -2 & 0 \\
0 & 0 & -2
\end{bmatrix}. \]
5) [20 points] Let
\[
\mathbf{v}_1 = (4, -4, 2, 2, 4, 1, 17), \\
\mathbf{v}_2 = (-1, 1, -1, 1, -1, -1, -6), \\
\mathbf{v}_3 = (3, -3, 2, 0, 3, 1, 14), \\
\mathbf{v}_4 = (10, -10, 5, 5, 10, 3, 43), \\
\mathbf{v}_5 = (2, -2, 1, 1, 2, 1, 9),
\]
and \( S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\} \), and \( V = \text{span}(S) \). Given that
\[
\begin{bmatrix}
\mathbf{v}_1 \\
\mathbf{v}_2 \\
\mathbf{v}_3 \\
\mathbf{v}_4 \\
\mathbf{v}_5 
\end{bmatrix} \sim \begin{bmatrix}
1 & -1 & 0 & 2 & 1 & 0 & 3 \\
0 & 0 & 1 & -3 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 0 & 3 & 1 \\
0 & 1 & 0 & -1 & -1 \\
0 & 0 & 1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 
\end{bmatrix},
\]
answer the questions below. [You do not need to justify any of the items below.]

(a) [5 points] What are the dimension of \( V \) and \( V^\perp \) [the orthogonal complement of \( V \) in \( \mathbb{R}^7 \)]?

\textit{Solution.} The dimension of \( V \) is the number of leading ones in either matrix in echelon form above, so it is 3.

The dimension of \( V^\perp \) is the number of columns without leading ones in the first matrix, so it is 4.

(b) [5 points] Find a basis for \( V \).

\textit{Solution.} We can take either the first three rows of the first matrix in echelon form, or \( \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \) [using the second matrix in echelon form].
(c) [4 points] Find a basis of $V^\perp$.

**Solution.** The basis can be found by finding a basis for the nullspace of the first matrix:

$$
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7 
\end{bmatrix} = \begin{bmatrix}
r - 2s - t - 3u \\
r \\
3s - 2u \\
s \\
t \\
-u \\
u 
\end{bmatrix} = r \begin{bmatrix}
1 \\
1 \\
0 \\
0 \\
0 \\
0 \\
0 
\end{bmatrix} + s \begin{bmatrix}
-2 \\
0 \\
3 \\
1 \\
0 \\
0 \\
0 
\end{bmatrix} + t \begin{bmatrix}
-1 \\
0 \\
0 \\
0 \\
1 \\
0 \\
0 
\end{bmatrix} + u \begin{bmatrix}
-3 \\
0 \\
0 \\
-2 \\
0 \\
-1 \\
1 
\end{bmatrix}.
$$

The column vectors in evidence above form the desired basis.

(d) [5 points] If possible, find a non-trivial linear combination [i.e., not all coefficients equal to zero] of the elements of $S$ which give the zero vector of $\mathbb{R}^7$. [**Hint:** Start by writing a vector of $S$ as a linear combination of the others.]

**Solution.** Using the second matrix in echelon form, and denoting its columns by $c_1$ to $c_5$, we can easily see that $c_5 = c_1 - c_2 - c_3$. Thus, $v_5 = v_1 - v_2 - v_3$. Thus,

$$1 \cdot v_1 + (-1) \cdot v_2 + (-1) \cdot v_3 + 0 \cdot v_4 + (-1) \cdot v_5 = 0$$

(e) [5 points] Which vectors from the standard basis of $\mathbb{R}^7$ you can add to the vectors in the basis of $V$ you’ve found above to obtain a basis of all of $\mathbb{R}^7$?

**Solution.** We just add standard basis vectors with leading ones in columns which have no leading ones in the first matrix in echelon form. So, we add $\{e_2, e_4, e_5, e_7\}$. 

\[\square\]