1) [15 points] Let $\vec{r}_1(t) = \langle t, t^2, t+1 \rangle$ and $\vec{r}_2(t) = \left\langle \sqrt{t}, t, t-1 \right\rangle$. Do these curves intersect? If so, in what point(s)?
[**Note:** I am *not* asking if they collide!]

Solution. We set:

$$t = \sqrt{s}$$
$$t^2 = s$$
$$t + 1 = s - 1$$

Substituting s from the second equation in the third:

$$t + 1 = t^2 - 1 \quad \Longrightarrow \quad t^2 - t - 2 = 0 \quad \Longrightarrow \quad t = -1, 2.$$

Since $\vec{r}_2(t)$ is only defined for $t \geq 0$ [as $t = \sqrt{s}$], we have only $t = 2$. So, there is only one point of intersection, being $\vec{r}_1(2) = \langle 2, 4, 3 \rangle = \vec{r}_2(4)$. □

2) [10 points] Let $\vec{r}(t) = \langle t\mathrm{e}^t, \cos(\pi t) \rangle$. Find a parametrization for the tangent line at the point given by $t = 0$.

Solution. We have $\vec{r}\,'(t) = \langle \mathrm{e}^t + t\mathrm{e}^t, -\pi \sin(\pi t) \rangle$. So, $\vec{r}\,'(0) = \langle 1, 0 \rangle$. Since $\vec{r}(0) = \langle 0, 1 \rangle$, we have that the tangent line is given by:

$$\vec{L}(t) = \langle 0, 1 \rangle + t \, \langle 1, 0 \rangle = \langle t, 1 \rangle \, .$$

□

3) [15 points] Compute the arc length of $\vec{r}(t) = \langle t^2, t^3 - 1\rangle$ for $0 \leq t \leq 1$.
[**Note:** You can leave numerical computations indicated.]

Solution. We have that $\vec{r}\,'(t) = \langle 2t, 3t^2\rangle$. Then,

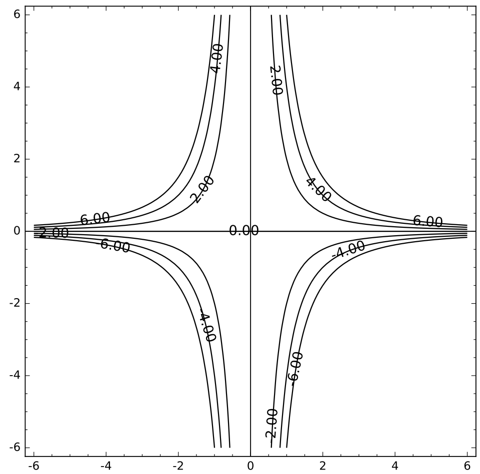
$$\begin{aligned}
\int_0^1 \|\vec{r}\,'(t)\|\,\mathrm{d}t &= \int_0^1 \sqrt{4t^2 + 9t^4}\,\mathrm{d}t \\
&= \int_0^1 t\sqrt{9t^2+4}\,\mathrm{d}t \qquad [\text{use } u = 9t^2 + 4] \\
&= \int_4^{13} \sqrt{u}\,\frac{1}{18}\,\mathrm{d}u \\
&= \frac{1}{18}\left[\frac{2u^{3/2}}{3}\right]_4^{13} \\
&= \frac{1}{27}[13^{3/2} - 4^{3/2}].
\end{aligned}$$

□

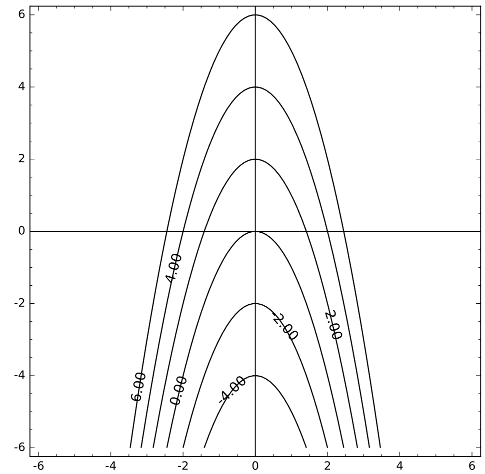
4) [15 points] Match the functions to their level curves by writing the corresponding letter inside the square containing the level curves. (Leave the level curves with no match unmarked.)

A: $f(x, y) = x^2 + y$ B: $f(x, y) = x/y^2$

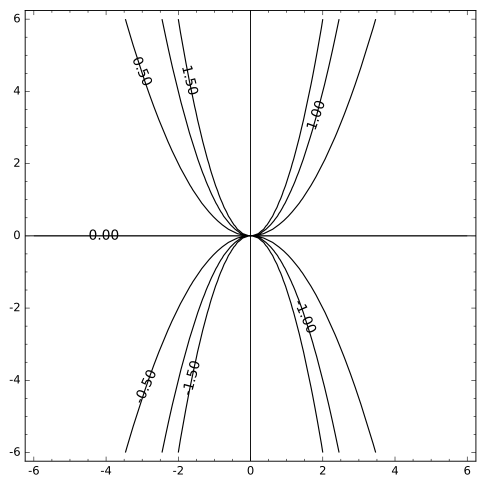
C: $f(x, y) = x^2 y$ D: $f(x, y) = x^2 - y^2$

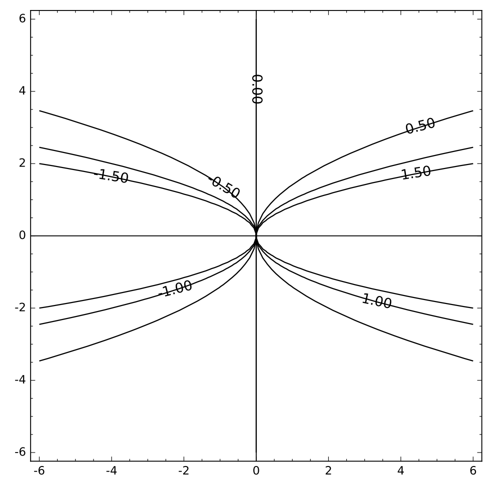


C

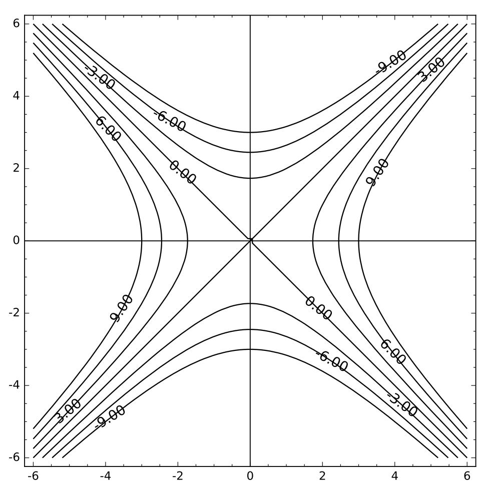


A

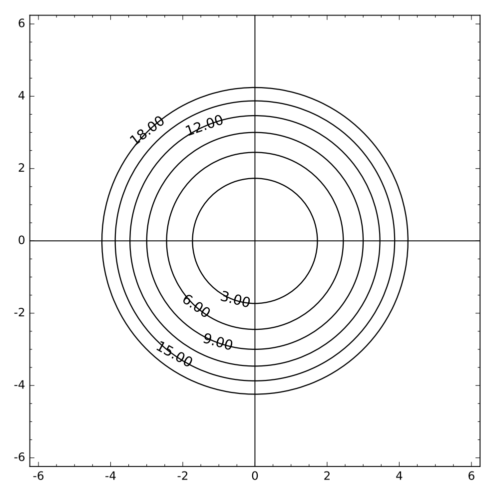




B



D



5) [15 points] Show that

$$\lim_{(x,y)\to(0,0)} \frac{xy - 2y^2}{x^2 + y^2}$$

does not exist.

[**Note:** You have to show your work *and* explain why your work shows that the limit does not exist.]

Solution. We compute the limit along lines through the origin $\langle t, mt \rangle$:

$$\lim_{t\to 0} \frac{mt^2 - 2m^2t^2}{t^2 + m^2t^2} = \lim_{t\to 0} \frac{m - 2m^2}{1 + m^2} = \frac{m - 2m^2}{1 + m^2}.$$

So, for $m = 0$ we get limit 0 and for $m = 1$ we get limit $-1/2$. Since these are different, the original limit does not exist. □

6) Compute:

(a) [5 points] $\lim_{(x,y)\to(1,2)} \frac{x^2 - y^3}{x+y}$

Solution. The function is continuous at $(1,2)$. So:

$$\lim_{(x,y)\to(1,2)} \frac{x^2 - y^3}{x+y} = \frac{1^2 - 2^3}{1+2} = -\frac{7}{3}.$$

□

(b) [10 points] $\lim_{(x,y)\to(0,0)} \ln(1 + 2x^2 + y^2) \cdot \sin\left(\frac{1}{x^2+y^2}\right)$

Solution. We have that $\left|\sin\left(\frac{1}{x^2+y^2}\right)\right| \leq 1$. So:

$$0 \leq \left|\ln(1 + 2x^2 + y^2) \cdot \sin\left(\frac{1}{x^2+y^2}\right)\right| \leq \left|\ln(1 + 2x^2 + y^2)\right|.$$

Since $\lim_{(x,y)\to(0,0)} \ln(1 + 2x^2 + y^2) = 0$, by the *Squeeze Theorem* we have that

$$\lim_{(x,y)\to(0,0)} \left|\ln(1 + 2x^2 + y^2) \cdot \sin\left(\frac{1}{x^2+y^2}\right)\right| = 0,$$

and so

$$\lim_{(x,y)\to(0,0)} \ln(1 + 2x^2 + y^2) \cdot \sin\left(\frac{1}{x^2+y^2}\right) = 0,$$

□

7) Let $f(x, y) = x^3 - xy^2 + y - 2$

(a) [5 points] Compute f_x and f_y.

Solution.

$$f_x(x, y) = 3x^2 - y^2$$

$$f_y(x, y) = -2xy + 1.$$

□

(b) [5 points] Give the equation of the tangent plane for $(x, y) = (0, 0)$.

Solution.

$$z = f_x(0, 0) \cdot (x - 0) + f_y(0, 0) \cdot (y - 0) + f(0, 0)$$

So:

$$z = y - 2.$$

□

(c) [5 points] Use linear approximation to approximate $f(-0.1, 0.2)$.

Solution. We have:

$$f(-0.1, 0.2) \approx 0.2 - 2 = -1.8.$$

□