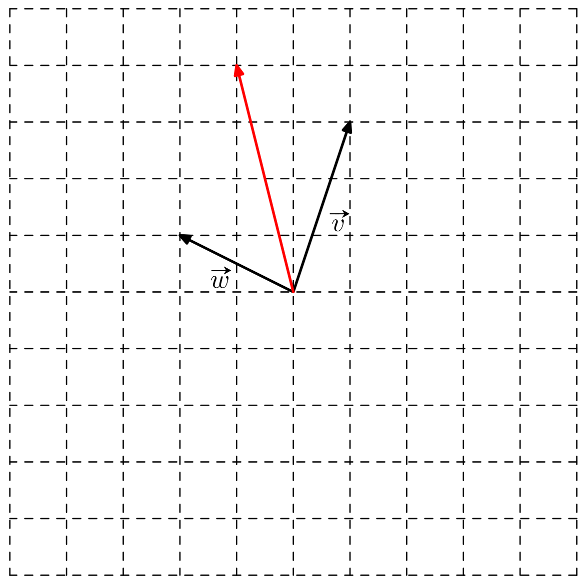
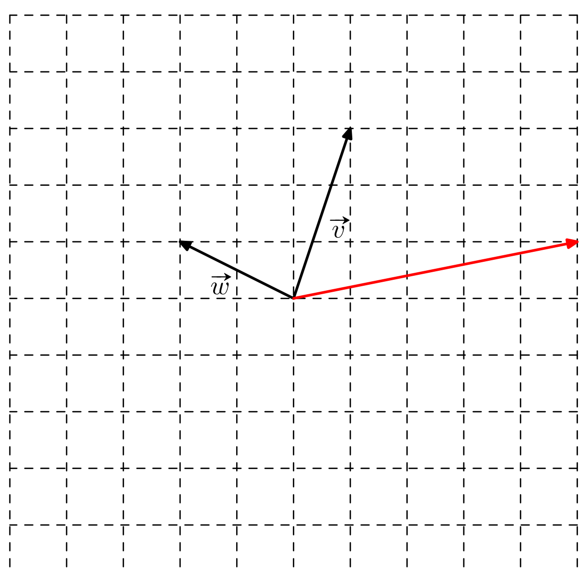
1) Vectors: I will use $\vec{v}$ for the vectors [as in class] instead of $\mathbf{v}$ [as in the book].

(a) [7 points] Draw the sum of the vectors $\vec{v}$ and $\vec{w}$ in the grid below.



(b) [8 points] Draw the $\vec{v} - 2\vec{w}$ in the grid below.



2) [10 points] Give the vector parametrization of the line passing through the points $P = (1, 0, -1)$ and $Q = (2, 2, 1)$.

Solution. So, P is a point on the line and its direction is $\overrightarrow{PQ} = \vec{Q} - \vec{P} = \langle 2-1, 2-0, 1-(-1)\rangle = \langle 1, 2, 2\rangle$. So:

$$\vec{r}(t) = \langle 1, 0, -1\rangle + t\,\langle 1, 2, 2\rangle \quad [\text{or } \vec{r}(t) = \langle 1+t, 2t, -1+2t\rangle\,.]$$

□

3) [15 points] Let $\vec{v} = \langle 1, -2, 0\rangle$ and $\vec{w} = \langle 1, 1, 1\rangle$. Compute $\vec{v}_{\parallel \vec{w}}$ [the projection of $\vec{v}$ along $\vec{w}$] and $\vec{v}_{\perp \vec{w}}$ [the orthogonal component of $\vec{v}$ with respect to $\vec{w}$].

Solution. We have:

$$\begin{aligned}\vec{v}_{\parallel \vec{w}} &= \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2}\,\vec{w} \\ &= \frac{1 \cdot 1 + (-2) \cdot 1 + 0 \cdot 1}{1^2 + 1^2 + 1^2}\,\langle 1, 1, 1\rangle \\ &= \left\langle -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\right\rangle\end{aligned}$$

Then:

$$\begin{aligned}\vec{v}_{\perp \vec{w}} &= \vec{v} - \vec{v}_{\parallel \vec{w}} \\ &= \langle 1, -2, 0\rangle - \left\langle -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\right\rangle \\ &= \left\langle \frac{4}{3}, -\frac{5}{3}, \frac{1}{3}\right\rangle.\end{aligned}$$

□

4) [10 points] Give the equation of the plane determined by $P = (0, 0, 3)$, $Q = (1, -1, -1)$, and $R = (1, -2, 0)$.

Solution. We have: $\overrightarrow{PQ} = \langle 1, -1, -4 \rangle$ and $\overrightarrow{PR} = \langle 1, -2, -3 \rangle$. Then,

$$\begin{aligned}
\vec{n} &= \overrightarrow{PQ} \times \overrightarrow{PR} \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & -4 \\ 1 & -2 & -3 \end{vmatrix} \\
&= (3 - 8)\,\vec{i} - (-3 + 4)\,\vec{j} + (-2 + 1)\,\vec{k} \\
&= \langle -5, -1, -1 \rangle\,.
\end{aligned}$$

So, $\vec{n} \cdot (\langle x, y, z \rangle - \vec{P}) = 0$, or

$$-5(x - 0) - (y - 0) - (z - 3) = 0 \quad [\text{or } 5x + y + (z - 3) = 0].$$

□

5) [10 points] Determine if the line given by $\vec{r}(t) = \langle 2 - t, 1 + 2t, 0 \rangle$ and the plane given by $2x - y - 3z = 5$ are either parallel, perpendicular, or neither. [For the sake of simplicity, we assume that a line on the plane is parallel to the plane.]

Solution. The normal vector to the plane is given by $\vec{n} = \langle 2, -1, -3 \rangle$ and the direction of the line is given by $\vec{v} = \langle -1, 2, 0 \rangle$. Since $\vec{n} \cdot \vec{v} = 2 \cdot (-1) + (-1) \cdot 2 + (-3) \cdot 0 = -4 \neq 0$, the line is not parallel to the plane.

If the line is perpendicular to the plane, then we need $\vec{v}$ to be a multiple of $\vec{n}$, which is not, as $\vec{v} = c\vec{n}$ means

$$\begin{aligned}
2c &= -1 & &[c = -1/2], \\
-c &= 2 & &[c = -2], \\
-3c &= 0 & &[c = 0].
\end{aligned}$$

So, no such c exists, and hence, it is not perpendicular either.
So, the line is neither parallel nor perpendicular to the plane. □

6) [15 points] Find the intersection of the planes $x - y + 2z = 3$ and $2x - z = 1$.

Solution. We solve the system:

$$\begin{array}{rrrcl} x & -y & +2z & = & 3 \\ 2x & & -z & = & 1. \end{array}$$

The second equation give $z = 2x - 1$. Substituting in the first, we get:

$$x - y + 2(2x - 1) = 3 \quad \Longrightarrow \quad y = 5x - 5.$$

Setting $x = t$, we get the line:

$$\vec{r}(t) = \langle t, 5t - 5, 2t - 1 \rangle .$$

□

7) [10 points] What is the volume of the parallelepipede determined by the vectors $\vec{u} = \langle 1, 0, 2 \rangle$, $\vec{v} = \langle 0, 2, 1 \rangle$, and $\vec{w} = \langle 1, -1, 0 \rangle$?

Solution. The volume is given by:

$$\begin{aligned} \left| \begin{vmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & -1 & 0 \end{vmatrix} \right| &= \left| 1 \cdot \begin{vmatrix} 2 & 1 \\ -1 & 0 \end{vmatrix} - 0 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + 2 \cdot \begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix} \right| \\ &= |(2 \cdot 0 - 1 \cdot (-1)) + 2(0 \cdot (-1) - 2 \cdot 1)| \\ &= |-3| = 3. \end{aligned}$$

□

8) [15 points] Give the cylindrical and spherical coordinates of the point $(x, y, z) = (3/2, \sqrt{3}/2, 1)$.

Solution. Cylindrical:

$$r = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{3}.$$

$$\theta = \arctan\left(\frac{\sqrt{3}/2}{3/2}\right) = \arctan(\sqrt{3}/3) = \frac{\pi}{6}.$$

$$z = 1.$$

Spherical:

$$\rho = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 + 1^2} = \sqrt{4} = 2.$$

$$\theta = \arctan\left(\frac{\sqrt{3}/2}{3/2}\right) = \arctan(\sqrt{3}/3) = \frac{\pi}{6}.$$

$$\phi = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}.$$

□