We have already talked about polynomials. I will stick with the “intuitive approach” [rather than the *formal* one].

Since we’ve talked about and used polynomials before, I will skip most of this section.

*Read the text (Section 4.5) if you are not comfortable with polynomials!*
**Theorem**

If $R$ is a *domain*, then for $f, g \in R[x]$, we have:

$$\deg(f \cdot g) = \deg(f) + \deg(g)$$

and

$$\deg(f + g) \leq \max\{\deg(f), \deg(g)\}.$$ 

*The equality in the expression above always hold if $\deg(f) \neq \deg(g)$.**

**Remember:** if $a \in R \setminus \{0\}$, then $\deg(a) = 0$ and $\deg(0) = -\infty$. 
Division Algorithm

We’ve already discussed the *division algorithm*: given $f, g \in R[x]$, with $g \neq 0$ and its leading coefficient [i.e., the coefficient of the term of highest degree] is a *unit*, then there are $q, r \in R[x]$ such that

$$f = g \cdot q + r,$$  
where $\deg(r) < \deg(g)$.

In particular, if $R$ is a field, and $g \neq 0$, then we have $q$ and $r$ as above.

The procedure to find $q$ and $r$ is exactly the same as the one you’ve learned for $\mathbb{R}[x]$ in algebra or precalculus.  
*If you forgot it, review!*
Division by Polynomial of Degree One

Let's work in \( R[x] \) where \( R \) is a domain. Dividing \( f(x) \) by \( g = x - a \), where \( a \in R \), we have that

\[
f = (x - a)q + r,
\]
where \( \deg(r) < \deg(x - a) = 1 \).

Hence, \( r \in R \) [a constant]. Evaluating at \( x = a \), we have

\[
f(a) = (a - a) \cdot q(a) + r(a) = r,
\]
i.e., \( r = f(a) \). So,

\[
f = (x - a)q + f(a), \quad \text{for some } q \in R[x].
\]

**Corollary**

If \( R \) is a domain and \( a \in R \), then \( (x - a) \) divides \( f \in R[X] \) iff \( f(a) = 0 \).
Corollary

If $R$ is a domain and $\deg(f) = n \geq 0$, then $f$ has at most $n$ roots in $R$.

Proof.

Proceed by induction. The case $n = 0$ is trivial.

Now, assume true for $(n - 1)$ and let $\deg(f) = n$. If $f$ has no roots in $R$, we are done. So, assume $a \in R$ and $f(a) = 0$. Then, $f = (x - a)g$, for some $g \in R[x]$. Then, $\deg(g) = n - 1$. By the IH, $g$ has at most $(n - 1)$ roots in $R$.

Now, we claim if $f(b) = 0$, for $b \in R$, then either $a = b$ or $g(b) = 0$: we have that $0 = f(b) = (b - a) \cdot g(b)$. Since $R$ is a domain [and $(b - a), g(b) \in R$], we have that either $b - a = 0$ or $g(b) = 0$, proving the claim and finishing the proof.
Other Remarks

Note that if $R$ is not a domain, the above result is not necessarily true: let $R = \mathbb{Z}/6\mathbb{Z}$ and $f = 2x$. Then $x = 0, 3$ are two distinct roots of $f$ [and $\deg(f) = 1$].

We have seen in class that $F[x]$, where $F$ is a field, is a PID [and hence noetherian and a UFD]. The idea is that we can do the long division by every non-zero element of $F[x]$. [Given $I$ and ideal of $F[x]$, take $f$ as an element of $I$ with least degree. Use long division to show that $I = (f)$.]

Note that we also have an Euclidean algorithm, exactly the same as for $\mathbb{Z}$, which allows us to explicitly write GCDs as linear combinations of elements. [Domains in which we have an “Euclidean Algorithm” are called Euclidean Domains. These are always PIDs, and hence also UFDs and noetherian.]