This is a take-home exam: You cannot talk to anyone (except me) about anything about this exam and you can only look at our book (Walker), class notes and solutions to our HW problems posted by me or done by yourself. No other reference, including the Internet. Failing to follow these instructions will result in a zero for the exam. Moreover, I will report the incident to the university and do all in my power to get the maximal penalty for the infraction.

Due date: noon on Wednesday (04/30). If you cannot bring it to my office (slide it under the door if I’m not in), a scanned/typed copy by e-mail would be OK.

1) [20 points] Let $F$ be a field of characteristic $p$ [and $p \neq 0$, a prime integer] and assume that $F$ is perfect [i.e., for each $a \in F$, there is a unique $b \in F$ such that $a = b^p$, i.e., each element of $F$ has a unique $p$-th root in $F$]. Let $\overline{F}$ be a fixed algebraic closure of $F$.

(a) Let $\alpha \in \overline{F}$ separable over $F$ and $\beta \in \overline{F}$ such that $\beta^p = \alpha$ [i.e., $\beta$ is the root of $x^p - \alpha$ in $\overline{F}$]. Show that $F[\beta] = F[\alpha]$. [Hint: Clearly $F[\beta] \supseteq F[\alpha]$. Let $f \overset{\text{def}}{=} \min_F(\alpha)$ and $g \overset{\text{def}}{=} \min_F(\beta)$. Use $f$ to find $g$, showing that $\deg f = \deg g$. Of course, you will need to use the fact that $F$ is perfect.]

(b) Prove that $F[\alpha]$ above is also perfect.

(c) Let $K$ be a finite separable extension of $F$. Show that $K$ is perfect.

[Note: It actually follows from the last item, and a little effort, that if $K/F$ is simply finite, then $K$ is perfect. But you don’t have to do it here.]

2) [20 points] Let $f = (x^4 - 1)(x^2 - 3) \in \mathbb{Q}[x]$. What is the Galois group isomorphic to? [Of course, justify! Note I am not asking for diagrams, intermediate fields, action on generators, etc. Only a description of the Galois group as a familiar group, such as $(\mathbb{Z}/5\mathbb{Z}) \times D_7 \times S_4$, or something like that.]

3) [20 points] Let $R$ be a commutative ring with 1, $A$, $B$ and $C$ be $R$-modules and $\phi : A \to B$ and $\psi : B \to C$ homomorphisms such that $\psi \circ \phi : A \to C$ is an isomorphism. Prove that $B = \text{im} \phi \oplus \ker \psi$. [Hint: Prove that if $b \in B$, then $\psi(b) = \psi(\phi(a))$ for some $a \in A$. What can we say about $b - \phi(a)$? Note that $\phi(a) \in \text{im} \phi$, of course.]
4) [20 points] Let $R$ be a PID, $M$ an $R$-module such that $M = T_1 \oplus F_1 = T_2 \oplus F_2$, where $T_i$ is torsion and $F_i$ is free. [Both are internal direct sums.]

(a) Prove that $T_1 = T_2$ [this is an equality!] and $F_1 \cong F_2$ [this is an isomorphism].

(b) Give an example where $F_1 \cong F_2$, but $F_1 \neq F_2$. [Hint: You can let $R = \mathbb{Z}$, $M = (\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}$, $T_1 = T_2 = (\mathbb{Z}/2\mathbb{Z}) \oplus \{0\}$, and $F_1 = \{0\} \oplus \mathbb{Z}$, and then find a $F_2$ that does the job, i.e., $F_2 \neq F_1$, $F_2 \cong \mathbb{Z}$ [hence cyclic], but $M = T_2 \oplus F_2$.]

5) [20 points] Let $V$ be a finite dimensional vector space over $F$ and $\phi : V \to V$ a linear transformation [i.e., homomorphism]. Then, we can give an $F[x]$-module structure to $V$ by,

$$(a_n x^n + \cdots + a_1 x + a_0) \cdot v \overset{\text{def}}{=} a_n \phi^n(v) + \cdots + a_1 \phi(v) + a_0 v$$

[where $\phi^n = \phi \circ \phi \circ \cdots \circ \phi$], in other words, $x \cdot v \overset{\text{def}}{=} \phi(v)$. [You don’t need to show that this indeed gives an $F[x]$-module structure.]

Assume that, as $F[x]$-modules, we have $V \cong F[x]/(f)$, for some $f \in F[x]$. Show that if $a \in F$ is such that $f(a) = 0$, then there exists $v \in V \setminus \{0\}$ such that $\phi(v) = a \cdot v$ [i.e., $a$ is an eigenvalue of $\phi$ and $v$ is an eigenvector of $\phi$ associated to $a$]. [Hint: $\phi(v) = a \cdot v$ iff $(x - a) \cdot v = 0$.]

[Note: The converse is also true.]

6) [20 points] Let $V$ be a finite dimensional vector space, $\{v_1^*, \ldots, v_n^*\}$ and basis of $V^*$ and $\{v_1^{**}, \ldots, v_n^{**}\}$ its dual basis [so, a basis of $V^{**}$]. Let also

$$\eta : V \to V^{**}$$

the natural isomorphism [i.e., $\eta(v)(v^*) \overset{\text{def}}{=} v^*(v)$, i.e., $\eta(v)$ is the evaluation at $v$ map].

(a) Prove that the original basis $\{v_1^*, \ldots, v_n^*\}$ is the dual basis of $\{v_1, \ldots, v_n\}$, where $v_i \overset{\text{def}}{=} \eta^{-1}(v_i^{**})$.

(b) Let $v_1^*, v_2^* \in V^* \setminus \{0\}$. Prove that if $\ker v_1^* = \ker v_2^*$, then $v_2^* = av_1^*$ for some $a \in F$ [i.e., $\{v_1^*, v_2^*\}$ is linearly dependent]. [Hint: Assume linearly independent and use part (a).]