Midterm 1

This is a take-home exam: You cannot talk to *anyone* (except me) about *anything* about this exam and you can only look at *our* book (Walker), class notes and solutions to *our* HW problems *posted by me* or done by yourself. No other reference, including the Internet. Failing to follow these instructions will result in a zero for the exam. Moreover, I will report the incident to the university and do all in my power to get the maximal penalty for the infraction.

Due date: noon on Friday (02/21). If you cannot bring it to class or to me, a scanned/typed copy by e-mail would be OK.

1) [20 points] Let R be a PID and I be an ideal of R. Prove that every ideal of R/I is principal. [In particular, if I is a prime ideal, then R/I is also a PID.]

2) [20 points] Let R be a commutative ring with 1 with no non-zero nilpotent element. [So, in R, if $a^n = 0$ for some $n \in \mathbb{Z}_{>0}$, then $a = 0$]. Prove that if $f \in R[x]$ is a zero divisor in $R[x]$, then there exists $b \in R \setminus \{0\}$ such that $b \cdot f = 0$. [Note I said "$b \in R \setminus \{0\}$", not "$b \in R[x] \setminus \{0\}$".]

3) [20 points] Prove that the quotient of a UFD by a prime ideal might not be a UFD. [**Hint:** We don't know many non-UFDs, so take a look at those!]

4) [20 points] Let F, K_1, K_2 and L be fields with $F \subseteq K_i \subseteq L$ for $i = 1, 2$.

(a) Prove that the intersection of all subfields of L containing both K_1 and K_2 is a field. [This field is called the *compositum of* K_1 *and* K_2 and it is denoted by $K_1 \cdot K_2$ or $K_1\, K_2$. It is clearly the minimal common extension of K_1 and K_2.]

(b) Prove that $K_1 \cdot K_2$ is the set of all $f(\alpha_1, \ldots, \alpha_k)$, with $f \in F(x_1, \ldots, x_k)$, for some $k \in \mathbb{Z}_{>0}$, defined at $(\alpha_1, \ldots, \alpha_k)$ [i.e., the denominator of the rational function $f(x_1, \ldots, x_k)$ does not vanish at $(\alpha_1, \ldots, \alpha_k)$] and $\alpha_i \in K_1 \cup K_2$ for all i.

(c) Prove that if K_1 and K_2 are both algebraic over F, then $K_1 \cdot K_2$ [as above] is also algebraic over F.

5) [20 points] Let p be a prime, $q = p^r$ for some $r \in \mathbb{Z}_{>0}$, and $\mathbb{F}_q$ be the finite field with q elements [in some fixed algebraic closure of $\mathbb{F}_p$]. Prove that if $\sigma \in \mathrm{Aut}(\mathbb{F}_q)$, then there exists some $t \in \mathbb{Z}_{>0}$ such that $\sigma(\alpha) = \alpha^t$ for all $\alpha \in \mathbb{F}_q$ and $\gcd(t, q-1) = 1$. [It is true, in fact, that t must be a power of p, but you don't need to show that.]