This is a take-home exam: You cannot talk to anyone (except me) about anything about this exam and you can only look at our book (Bona) and class notes. No other reference is allowed, including the Internet. Failing to follow these instructions will result in a zero for the exam. Moreover, I will report the incident to the university and do all in my power to get the maximal penalty for the infraction.

As you will have a lot of time, I expect the solutions to be well written and clearly explained. If you need clarifications on any statement, please use the “Q&A (Math Related)” forum on Blackboard.

Due date: 5pm on Monday (05/05) by e-mail. Please send as a PDF and make sure your scanned/typed exam is clear and legible. If I can’t read your solution, I will give you zero for the problem!

Always show work!

1) [12 points] A student has 5 different combinatorics books, 8 different algebra books, 4 different analysis books and 6 different geometry books. In how many ways can these books be arranged on a shelf so that all the books of the same topic are together?

2) [12 points] How many rearrangements of the alphabet [26 letters] are such that the vowels, A, E, I, O and U, do not appear in order. [Note that there can be letters between the vowels. But we don’t want for A to come before E, then E before I, etc.]

3) [12 points] A student has to work 12 hours a week [7 days]. If he can only work for an integer number of hours a day [so “one and a half hour on Monday” is not allowed], he does not need to work every single day [so, zero hours in a day is allowed], but he has to work on Mondays, Wednesdays and Fridays [for at least one hour]. In how many ways can he plan a work week? [Note that the times when he works is irrelevant, we only car about how many hours he works on each day of the week.]
4) [12 points] Show that
\[ \sum_{i=0}^{20} \sum_{j=0}^{20-i} \frac{n!}{i! \cdot j! \cdot (n-i-j)!} \cdot 2^i \cdot 3^j = 6^{20}. \]

[Hint: \( i + j + (n - i - j) = n \).]

5) [13 points] In how many permutations on \( n \)-elements \( [n \geq 4] \) do we have 1 and 2 in the same cycle, but 3 and 4 not in the same cycle?

6) [13 points] How many arrangements of \((1, 1, 2, 2, 3, 3, 4, 5, 6, 7)\) have no pair of equal digits in consecutive spots.

7) [13 points] Let \( a_n \), for \( n \geq 2 \), be the numbers of sequences of 0’s and 1’s with at least one instance of two consecutive 0’s. Find a recurrence relation for the \( a_n \)'s. [That means, a formula for \( a_n \) which depends on the previous \( a_i \)'s. For instance, something like \( a_n = a_{n-1} + 2n \) or \( a_n = 3a_{n-1} - 2a_{n-3} \).]

8) [13 points] Let \( a_n \) be such that \( a_0 = 1 \) and \( a_n = 2a_{n-1} + n \). Find a closed formula [no summation] for \( a_n \).