Section 4.1 Homework

9. $S \rightarrow$ surface area of the snowball (cm^2)
$x \rightarrow$ diameter of the snowball (cm
$t \rightarrow$ time (min)

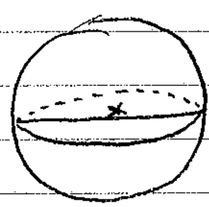
(a) Given: We are given that the surface area of the snowball is decreasing at a rate of $1\ cm^2/min$. In other words

$$\frac{dS}{dt} = -1\ cm^2/min$$

(b.) Unknown: We want the rate at which the diameter is decreasing when the diameter is 10 cm.
i.e.

$$\frac{dx}{dt} \quad \text{when} \quad x = 10\ cm$$

(c)



We assume the snowball is a sphere.

(d) $S = 4\pi r^2$ where r is the radius.

$$r = \tfrac{1}{2}x \implies S = 4\pi\left(\tfrac{1}{2}x\right)^2$$

$$\implies S = \pi x^2$$

$$\implies \frac{d}{dt}(S) = \frac{d}{dt}\left(\pi x^2\right)$$

$$\implies \frac{dS}{dt} = 2\pi x \frac{dx}{dt}$$

(e) $-1 = 2\pi(10)\frac{dx}{dt} \implies \frac{dx}{dt} = -\frac{1}{20\pi}$

The diameter is decreasing at a rate of $-\frac{1}{20\pi}$ cm/min when the diameter is 10 cm.

11) $x \rightarrow$ horizontal distance from the plane to a point directly above the station (mi)

$y \rightarrow$ distance from the plane to the station (mi)

$t \rightarrow$ time (h)

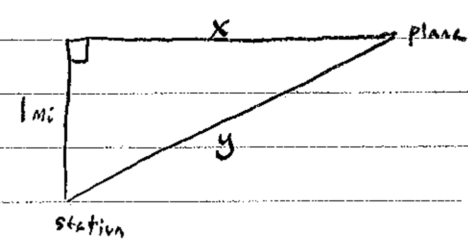
(a) <u>Given</u>: The speed of the plane is 500 mi/h

i.e. $\frac{dx}{dt} = 500$ mi/h

(b) <u>Unknown</u>: The rate at which the distance between the plane and station are increasing when the plane is 2 mi away.

i.e. $\frac{dy}{dt}$ when $y = 2$

(c)



(d) Use pythagorean theorem

$$y^2 = 1 + x^2$$

$$\Rightarrow \frac{d}{dt}\left(y^2\right) = \frac{d}{dt}\left(1 + x^2\right)$$

$$\Rightarrow 2y\frac{dy}{dt} = 2x\frac{dx}{dt}$$

$$\Rightarrow \frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt}$$

(e) When $y = 2$ $x = ?$

$$2^2 = x^2 + 1^2$$

$$x^2 = 3$$

$$x = \sqrt{3}$$

So by substituting our known values into $\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt}$

$$\frac{dy}{dt} = \frac{\sqrt{3}}{2}(500) = 250\sqrt{3}$$

The distance between the plane and station is increasing at a rate of $250\sqrt{3}$ mi/h.

12) $x \to$ distance from the man to the pole (ft)
$y \to$ distance from the pole to the tip of his shadow (ft)
$t \to$ time (s)

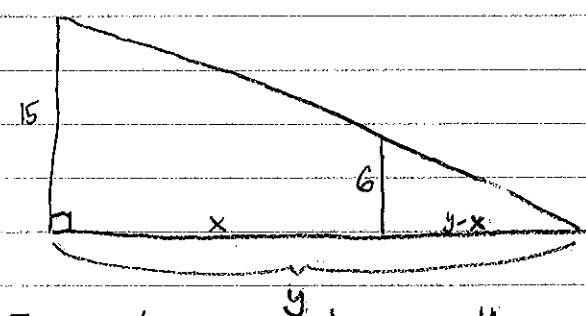
(a) <u>Given</u>: The speed the man is walking away from the pole is 5 ft/s

i.e. $\frac{dx}{dt} = 5 \text{ ft/s}$

(b) <u>Unknown</u>: The speed the tip of the shadow is moving when he is 40 ft from the pole

i.e. $\frac{dy}{dt}$ when $x = 40$

(c)



(d) The large and small triangles are similar and thus

$$\frac{y}{15} = \frac{y-x}{6}$$

$$\Rightarrow 6y = 15y - 15x$$

$$\Rightarrow 9y = 15x$$

$$\Rightarrow y = \frac{5}{3}x$$

$$\Rightarrow \frac{dy}{dt} = \frac{5}{3}\frac{dx}{dt}$$

(e) $\frac{dy}{dt} = \frac{5}{3}(5) = \frac{25}{3}$

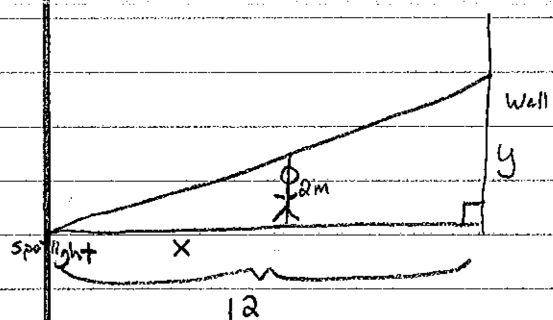
The tip of the shadow is moving at a speed of $25/3$ ft/s.

14) $x \rightarrow$ distance from the man to the spotlight (m)
$y \rightarrow$ height of his shadow (m)
$t \rightarrow$ time (s)

Given: $\frac{dx}{dt} = 1.6$ m/s

Unknown: $\frac{dy}{dt}$ when $x = 8$ m

(the wall is 12 m from the spotlight and the man is 4m from the wall so $x = 12 - 4 = 8$)



The large and small triangles are similar

$$\Rightarrow \frac{y}{12} = \frac{2}{x} \Rightarrow y = \frac{24}{x}$$

$$\Rightarrow \frac{d}{dt}\left(y\right) = \frac{d}{dt}\left(\frac{24}{x}\right)$$

$$\Rightarrow \frac{dy}{dt} = -\frac{24}{x^2}\frac{dx}{dt}$$

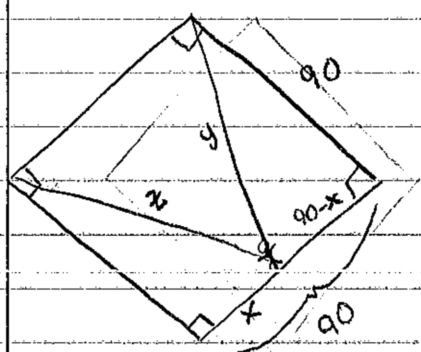
So when $x = 8$ m

$$\frac{dy}{dt} = -\frac{24}{8^2}(1.6) = \frac{-24}{64}\cdot\frac{16}{10} = -\frac{6}{10} = -0.6$$

The shadow is decreasing at a rate of 0.6 m/s when he is 4m from the building.

16

$x \to$ distance the runner is from home (ft)
$y \to$ distance from the runner to 2nd (ft)
$z \to$ distance from the runner to 3rd (ft)
$t \to$ time (s)



(a) <u>Given:</u> $\frac{dx}{dt} = 24 \text{ ft/s}$

<u>Unknown:</u> $\frac{dy}{dt}$ when $x = 45$ ft

By Pythagorean theorem

$$y^2 = (90-x)^2 + 90^2 \Rightarrow \frac{d}{dt}\left(y^2\right) = \frac{d}{dt}\left((90-x)^2 + 90^2\right)$$

$$\Rightarrow 2y\frac{dy}{dt} = 2(90-x)\frac{d}{dt}(90-x)$$

$$\Rightarrow 2y\frac{dy}{dt} = -2(90-x)\frac{dx}{dt}$$

$$\Rightarrow \frac{dy}{dt} = -\frac{1}{y}(90-x)\frac{dx}{dt}$$

When $x = 45$ ft $y = ?$

$$y^2 = 45^2 + 90^2 = 10125$$

$$\Rightarrow y = \sqrt{10125} = 45\sqrt{5}$$

$$45^2 + (2(45))^2 = 5(45)^2$$

$$\frac{dy}{dt} = -\frac{1}{45\sqrt{5}}(90-45)(24) = -\frac{24}{\sqrt{5}}$$

So the distance from the runner to 2nd is decreasing at a rate of $24/\sqrt{5}$ ft/s when he is 45 ft from home and running to 1st.

(b.) Unknown $\frac{dz}{dt}$ when $x = 45$

$$z^2 = x^2 + 90^2 \Rightarrow \frac{d}{dt}(z^2) = \frac{d}{dt}(x^2 + 90^2)$$

$$\Rightarrow 2z\frac{dz}{dt} = 2x\frac{dx}{dt} \qquad \Rightarrow \frac{dz}{dt} = \frac{x}{z}\frac{dx}{dt}$$

When $x = 45$ ft $\quad z = ?$

$$z^2 = \sqrt{45^2+90^2} = 45\sqrt{5}$$

So

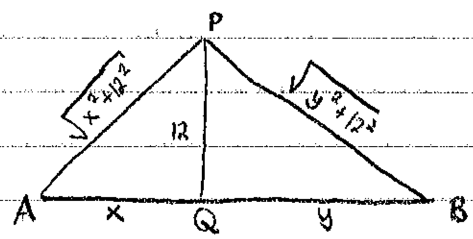
$$\frac{dz}{dt} = \frac{45}{45\sqrt{5}}(24) = \frac{24}{\sqrt{5}}$$

Thus when the runner is 45 ft to 1st the distance from the runner to 3rd is increasing at a rate of $24/\sqrt{5}$ ft/s.

24) $x \to$ distance from cart A to Q (ft

$y \to$ distance from cart B to Q (ft)

$t \to$ time (s)



<u>Given:</u> $\frac{dx}{dt} = -2$ ft/s

<u>unknown:</u> $\frac{dy}{dt}$ when $x = 5$ ft

$$\sqrt{x^2+12^2} + \sqrt{y^2+12^2} = 39 \Rightarrow \frac{d}{dt}\left((x^2+144)^{1/2} + (y^2+144)^{1/2}\right) = \frac{d}{dt}(39)$$

$$\Rightarrow \frac{1}{2}(x^2+144)^{-1/2}\left(2x\frac{dx}{dt}\right) + \frac{1}{2}(y^2+144)^{-1/2}\left(2y\frac{dy}{dt}\right) = 0$$

$$\Rightarrow -\frac{y}{\sqrt{y^2+144}}\frac{dy}{dt} = \frac{x}{\sqrt{x^2+144}}\frac{dx}{dt}$$

$$\Rightarrow \frac{dy}{dt} = -\frac{x\sqrt{y^2+144}}{y\sqrt{x^2+144}}\frac{dx}{dt}$$

When $x = 5$ ft $\quad y = ?$

$$\sqrt{(5)^2+144} + \sqrt{y^2+144} = 39 \Rightarrow 13 + \sqrt{y^2+144} = 39$$

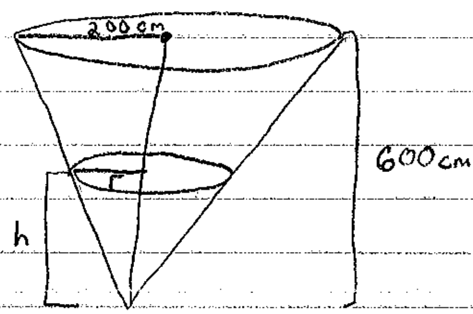
$$\Rightarrow \sqrt{y^2+144} = 26 \Rightarrow y^2 + 144 = 26^2$$

$$\Rightarrow y^2 = 532 \Rightarrow y = \sqrt{532} = 2\sqrt{133}$$

$$\frac{dy}{dt} = -\frac{5\sqrt{\sqrt{532}^2+144}}{2\sqrt{133}\sqrt{5^2+144}}(2) = -\frac{5(26)}{13\sqrt{133}} = \frac{-10}{\sqrt{133}}$$

Cart B is moving to Q at a rate of $-10/\sqrt{133}$ ft/s.

22)
$h \to$ height of the water level (cm)
$r \to$ radius of the cylinder at the water level (cm)
$V \to$ volume of water in the tank (cm^3)
$t \to$ time (min)



Given: $\frac{dV}{dt} = C - 10{,}000$ ← flowing out at a rate of $10{,}000\ cm^3/min$

↑ flowing in at a constant rate

$\frac{dh}{dt} = 20\ cm/min$ when $h = 200$ cm

Unknown: $C \to$ rate at which the water is flowing into the tank.

$V = \frac{1}{3}\pi r^2 h$, $\frac{r}{h} = \frac{200}{600} \Rightarrow 6r = 2h$ (similar Δ's)

$\Rightarrow r = \frac{1}{3}h$

$\Rightarrow V = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h \Rightarrow V = \frac{1}{27}\pi h^3$

$\Rightarrow \frac{dV}{dt} = \frac{1}{9}\pi h^2 \frac{dh}{dt}$

So when $h = 200$ cm

$\frac{dV}{dt} = \frac{1}{9}\pi(200)^2(20) = \frac{800{,}000}{9}\pi$

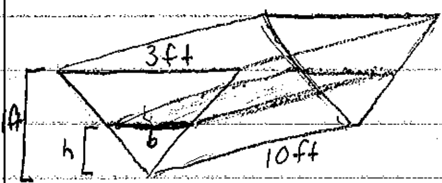
Now $\frac{dV}{dt} = C - 10{,}000$ so

$$C = \frac{800{,}000}{9}\pi + 10{,}000 \approx 289{,}253 \text{ cm}^3/\text{min}$$

Therefore the water is flowing into the tank at approximately $289{,}253 \text{ cm}^3/\text{min}$.

23. $V \rightarrow$ volume of water in the trough (ft^3)

$h \rightarrow$ height of the water level (ft)



<u>Given</u>: $\dfrac{dV}{dt} = 12\ ft^3/min$

<u>Unknown</u>: $\dfrac{dh}{dt}$ when $h = \frac{1}{2}\ ft$

$V = \frac{1}{2} b h (10)$ $\qquad \dfrac{b}{h} = \dfrac{3}{1} \Rightarrow b = 3h$

(similar Δ's)

$\Rightarrow V = \frac{1}{2}(3h)h(10)$

$\Rightarrow V = 15h^2 \quad \Rightarrow \quad \dfrac{dV}{dt} = 30h\dfrac{dh}{dt}$

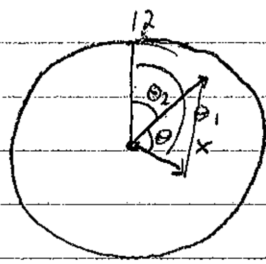
$\Rightarrow \quad \dfrac{dh}{dt} = \dfrac{1}{30h}\dfrac{dV}{dt}$

So when $h = \frac{1}{2}\ ft$

$$\frac{dh}{dt} = \frac{1}{15}(12) = \frac{4}{5}$$

Thus the height of the water level is rising at a rate of $4/5\ ft/min$.

38) $\theta_1 \rightarrow$ angle from 12 to the hour hand (radians)
$\theta_2 \rightarrow$ angle from 12 to the minute hand (radians)
$\theta \rightarrow$ angle between the minute and hour hands (rad)
$t \rightarrow$ time (h)



$x \rightarrow$ distance between the tips of the hour & minute hand (mm)

Unknown: $\frac{dx}{dt}$ when $\theta = \pi/6$

$\pi/6$ is the angle measure between every hour mark on the watch face b/c the circle has 12 equally spaced marks and has a total angle measure of 2π.

$$\theta = \theta_1 - \theta_2 \implies \frac{d\theta}{dt} = \frac{d\theta_1}{dt} - \frac{d\theta_2}{dt}$$

- The hour hand moves $\pi/6$ rad/h $\implies \frac{d\theta_1}{dt} = \pi/6$ rad/h
- The minute hand moves 2π rad/h $\implies \frac{d\theta_2}{dt} = 2\pi$ rad/h

So $$\frac{d\theta}{dt} = \pi/6 - 2\pi = -\frac{11\pi}{6} \text{ rad/h}$$

By the law of cosines

$$x^2 = 4^2 + 8^2 - 2(4)(8)\cos(\theta)$$

$$\implies 2x\frac{dx}{dt} = 64\sin(\theta)\frac{d\theta}{dt}$$

$$\Rightarrow \frac{dx}{dt} = \frac{32}{x}\sin(\theta)\frac{d\theta}{dt}$$

When $\theta = \pi/6$ what is x ?

$$x^2 = 4^2 + 8^2 - 2(4)(8)\cos(\pi/6)$$

$$\Rightarrow x = \sqrt{80 - 32\sqrt{3}}$$

Therefore when $\theta = \pi/6$

$$\frac{dx}{dt} = \frac{32}{\sqrt{80-32\sqrt{3}}}\sin(\pi/6)\left(-\frac{11\pi}{6}\right) = \frac{-88\pi}{3\sqrt{80-32\sqrt{3}}} \approx -18.6\,\text{mm/h}$$

So the distance between the tips of the hands is decreasing at a rate of 18.6 mm/h at one o'clock.