1) [10 points] Let \( \mathbf{u} = (1, 1) \) and \( \mathbf{v} = (3, k) \). Find \( k \) such that the angle between \( \mathbf{u} \) and \( \mathbf{v} \) is \( \pi/4 \).

**Solution.** We have:

\[
(1, 1) \cdot (3, k) = \| (1, 1) \| \| (3, k) \| \cos(\pi/4).
\]

So,

\[
3 + k = \sqrt{2} \sqrt{9 + k^2} \sqrt{2}/2 = \sqrt{k^2 + 9}
\]

Squaring:

\[
k^2 + 6k + 9 = k^2 + 9,
\]

and hence \( 6k = 0 \), i.e., \( k = 0 \).

Now, since we’ve squared the equation, we need to check that we’ve got it right [as squaring “1 = −1” gives a true statement], but indeed,

\[
\frac{(1, 1) \cdot (3, 0)}{\sqrt{2} \cdot 3} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \cos(\pi/4).
\]

[To see why this is necessary, replace \((3, k)\) by \((-3, k)\). The algebra will give you \( k = 0 \), but the fact is that there is no \( k \) that will make the angle \( \pi/4 \).]

[Note that this could be easily seen geometrically.]
2) [20 points] Give the matrix that represents the following linear transformation (from $\mathbb{R}^3$ to $\mathbb{R}^3$): projection to the $xy$-plane, followed by a rotation of $\pi/2$ around the $z$-axis, followed by a reflection on the $yz$-plane.

Solution. We find the matrix by checking what it does to the standard basis vectors $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$.

$e_1 \rightarrow e_1 \rightarrow e_2 \rightarrow e_2$, 

also, 

$e_2 \rightarrow e_2 \rightarrow -e_1 \rightarrow e_1$, 

and finally, 

$e_3 \rightarrow 0 \rightarrow 0 \rightarrow 0$.

So, the matrix is:

$$\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}. $$

Alternatively, the matrices for the three transformations are, respective:

$$\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad \begin{bmatrix}
\cos(\pi/2) & -\sin(\pi/2) & 0 \\
\sin(\pi/2) & \cos(\pi/2) & 0 \\
0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad \begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix},$$

and hence the matrix of the linear transformation we are looking for is:

$$\begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}. $$
3) [15 points] Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $T(2, 1) = (3, 0)$ and $T(0, 1) = (1, -2)$. Find the inverse of $T$ or show that such inverse does not exist.

Solution. We have that the first column of $[T]$ is $T(1, 0)$, while the second column is $T(0, 1)$. Hence, the statement gives us the second column. To find the first, we have [since $T$ is linear], we have that

$$(3, 0) = T(2, 1) = T(2(1, 0) + (0, 1)) = 2T(1, 0) + T(0, 1) = 2T(1, 0) + (1, -2).$$

Solving for $T(1, 0)$, we have $T(1, 0) = 1/2 (2, 2) = (1, 1)$, and hence,

$$[T] = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}.$$

Then $T$ is invertible if, and only if, the matrix $[T]$ is invertible, which is true:

$$[T^{-1}] = [T]^{-1} = -\frac{1}{3} \begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix}.$$

Thus,

$$T^{-1}(x, y) = \left(\frac{2x + y}{3}, \frac{x - y}{3}\right).$$
4) [15 points] Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and assume that 0 is an eigenvalue of $T$. Can $T$ be one-to-one? Can it be onto? Justify your answers!

Proof. No [for both]. Saying that 0 is an eigenvalue is the same as to say that there is a vector $\mathbf{v} \neq \mathbf{0}$ such that $T(\mathbf{v}) = 0 \cdot \mathbf{v} = \mathbf{0}$. Hence, since always for a linear transformation $T(\mathbf{0}) = \mathbf{0}$, we have two different vectors [namely, $\mathbf{v}$ and $\mathbf{0}$] with the same value by $T$ [namely, $\mathbf{0}$], and hence $T$ is not one-to-one.

Now, since the domain and codomain of $T$ are equal [namely, $\mathbb{R}^n$], we have that $T$ is one-to-one if, and only if, it is onto. Hence, since $T$ is not one-to-one, it cannot be onto. $\square$
Let \( V = \mathbb{R}^2 \) with the usual sum of vectors in \( \mathbb{R}^2 \), but with the following multiplication by real numbers: \( k(x, y) = (ky, kx) \). Show that \( V \) is not a vector space.

**Solution.** We have
\[
1 (1, 0) = (0, 1),
\]
and thus, property 8 [from the list given below] fails. Hence, it cannot be a vector space.

[You could also show instead the property 7 fails:
\[
(2 \cdot 3) (1, 0) = 6 (0, 1) = (0, 6),
\]
while
\[
2 (3 (1, 0)) = 2 (0, 3) = (6, 0).
\]
But, you only need to show that one fails.]
6) [10 points] Consider the set of all matrices

\[
\begin{bmatrix}
a & 1 \\
2 & b \\
\end{bmatrix}, \quad a, b \in \mathbb{R}.
\]

with:

\[
\begin{bmatrix}
a & 1 \\
2 & b \\
\end{bmatrix} + \begin{bmatrix}
a' & 1 \\
2 & b' \\
\end{bmatrix} = \begin{bmatrix}
a + a' & 1 \\
2 & b + b' \\
\end{bmatrix}, \quad k \begin{bmatrix}
a & 1 \\
2 & b \\
\end{bmatrix} = \begin{bmatrix}
ka & 1 \\
2 & kb \\
\end{bmatrix}.
\]

This set with this sum and scalar multiplication is a vector space. [You do not need to prove it! Just take my word for it.] What is the zero of this vector space? What is 

\[-\begin{bmatrix}
3 & 1 \\
2 & -1 \\
\end{bmatrix}?\]

\[\text{Solution.}\]

If

\[0 = \begin{bmatrix}
x & 1 \\
2 & y \\
\end{bmatrix},\]

we must have that

\[
\begin{bmatrix}
a & 1 \\
2 & b \\
\end{bmatrix} + \begin{bmatrix}
x & 1 \\
2 & y \\
\end{bmatrix} = \begin{bmatrix}
a + x & 1 \\
2 & b + y \\
\end{bmatrix} = \begin{bmatrix}
a & 1 \\
2 & b \\
\end{bmatrix}.
\]

Thus, we must have \(x = y = 0\), i.e.,

\[0 = \begin{bmatrix}
0 & 1 \\
2 & 0 \\
\end{bmatrix}.
\]

If

\[-\begin{bmatrix}
3 & 1 \\
2 & -1 \\
\end{bmatrix} = \begin{bmatrix}
x & 1 \\
2 & y \\
\end{bmatrix},\]

we must have that

\[
\begin{bmatrix}
3 & 1 \\
2 & -1 \\
\end{bmatrix} + \begin{bmatrix}
x & 1 \\
2 & y \\
\end{bmatrix} = \begin{bmatrix}
3 + x & 1 \\
2 & -1 + y \\
\end{bmatrix} = 0 = \begin{bmatrix}
0 & 1 \\
2 & 0 \\
\end{bmatrix}.
\]

Thus, we must have \(x = -3\) and \(y = 1\), i.e.,

\[-\begin{bmatrix}
3 & 1 \\
2 & -1 \\
\end{bmatrix} = \begin{bmatrix}
-3 & 1 \\
2 & 1 \\
\end{bmatrix}.
\]

\[\square\]
7) [15 points] Let \( V \) be the set of all functions [from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \)] of the form \( f(x, y) = (ax + by, ax^2 + by^2) \), where \( a, b \in \mathbb{R} \). Is \( V \) a vector space with the usual sum and scalar multiplication of functions? [Justify!]

Solution. The set of functions from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \), since \( \mathbb{R}^2 \) is a vector space, is itself a vector space [which I denoted in class by \( F(\mathbb{R}^2, \mathbb{R}^2) \)]. Since \( V \) is inside this vector space [and we are using the same operations], it suffices to show that it is a subspace.

If \( f(x, y) = (ax + by, ax^2 + by^2) \) and \( g(x, y) = (a'x + b'y, a'x^2 + b'y^2) \), then

\[
    f(x, y) + g(x, y) = ((a + a')x + (b + b')y, (a + a')x^2 + (b + b')y^2),
\]

and hence \( f(x, y) + g(x, y) \in V \).

Also, given any \( k \in \mathbb{R} \),

\[
    kf(x, y) = ((ka)x + (kb)y, (ka)x^2 + (kb)y^2),
\]

and hence \( kf(x, y) \in V \).

These two properties gives us that \( V \) is then a vector space [more precisely, a subspace of \( F(\mathbb{R}^2, \mathbb{R}^2) \)]. [Note that \( V \) is not empty, as \( f(x, y) = (0, 0) \in V \).] 

\[\square\]
Vector Space Requirements

A non-empty set $V$ with a sum and a scalar product is a vector space if it satisfies the following conditions:

0. $u + v \in V$ for all $u, v \in V$, and $ku \in V$ for all $u \in V$ and $k \in \mathbb{R}$;

1. $u + v = v + u$ for all $u, v \in V$;

2. $(u + v) + w = u + (v + w)$ for all $u, v, w \in V$;

3. there is $0 \in V$ such that $0 + u = u$ for all $u \in V$;

4. given $u \in V$, there exists $-u \in V$ such that $u + (-u) = 0$;

5. $k(u + v) = ku + kv$ for all $u, v \in V$ and $k \in \mathbb{R}$;

6. $(k + l)u = ku + lu$ for all $u \in V$ and $k, l \in \mathbb{R}$;

7. $k(lu) = (kl)u$ for all $u \in V$ and $k, l \in \mathbb{R}$;

8. $1u = u$ for all $u \in V$. 
