Extra Credit 2

Math 456

May 2, 2007

1. Let F be a field and

$$F[[x]] \stackrel{\text{def}}{=} \left\{ \sum_{n=0}^{\infty} a_n x^n \;:\; a_n \in F \right\},$$

i.e., the *ring of power series over* F. This is indeed an *integral domain*, with the sum and product defined as expected:

$$\left[\sum_{n=0}^{\infty} a_n x^n\right] + \left[\sum_{n=0}^{\infty} b_n x^n\right] \stackrel{\text{def}}{=} \left[\sum_{n=0}^{\infty} (a_n + b_n) x^n\right]$$

and

$$\left[\sum_{n=0}^{\infty} a_n x^n\right] \cdot \left[\sum_{n=0}^{\infty} b_n x^n\right] \stackrel{\text{def}}{=} \left[\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n\right]$$

[*You don't have to prove any of the above!!*] Let $\sigma : F[[x]] - \{0\} \to \{0, 1, 2, \dots\}$ be defined as: $\sigma\left(\sum_{n=0}^{\infty} a_n x^n\right)$ is the *smallest* n such that $a_n \neq 0$.

In this problem we will prove that $F[[x]]$ is a Euclidean domain.

(a) Prove that $F[[x]]^{\times} = \{a \in F[[x]] \;:\; \sigma(a) = 0\}$.

Proof. Let $a = \sum_{i=0}^{\infty} a_i x^i \in F[[x]]^{\times}$ and $b = \sum_{i=0}^{\infty} b_i x^i$ such that $ab = 1$. Comparing coefficients [of the product with the coefficients of 1], we have:

$$\begin{cases} a_0 b_0 = 1 \\ a_0 b_1 + a_1 b_0 = 0 \\ a_0 b_2 + a_1 b_1 + a_2 b_0 = 0 \\ \vdots \\ a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0 = 0 \\ \vdots \end{cases}$$

So, by the first equation, it is clearly necessary that $a_0 \neq 0$.

Moreover, it is also sufficient! To see that, assume that $a_0 \neq 0$. To prove that $a \in F[[x]]^{\times}$, we need to solve the system of equations above where the a_i's are given and the b_i's are the unknowns. We proceed by induction on the number of equations for which we can find a solution.

For the first equation, since $a_0 \neq 0$ [and F is a field], we have that $b_0 = 1/a_0$. Now, assume we can find $b_0, \ldots, b_{n-1}$ solutions for the first n equations. Then, we can find a solution for the $(n+1)$-th equation, namely [since $a_0 \neq 0$] $b_n = -(a_1 b_{n-1} + \cdots + a_n b_0)/a_0$.

Therefore, $a \in F[[x]]^\times$ if, and only if, $a_0 \neq 0$.

□

(b) Prove that for all $a \in F[[x]]$, we can write $a = x^{\sigma(a)} a'$, where $a' \in F[[x]]^\times$.

Proof. By the definition of $\sigma(a)$, we can always write $a = \sum_{i=\sigma(a)}^{\infty} a_i x^i$. So, factoring the power $x^{\sigma(a)}$ from the summation, we have $a = x^{\sigma(a)} \sum_{i=\sigma(a)}^{\infty} a_i x^{i-\sigma(a)} = x^{\sigma(a)} \sum_{i=0}^{\infty} a_{i+\sigma(a)} x^i$. But, since $a_{\sigma(a)} \neq 0$ [by defintion of σ], we clearly have that $\sigma(\sum_{i=0}^{\infty} a_{i+\sigma(a)} x^i) = 0$, and hence it is in $F[[x]]^\times$. □

(c) Use the above to prove that $a \mid b$ in $F[[x]]$ iff $\sigma(a) \leq \sigma(b)$.

Proof. Note that for all $a, b \in F[[x]]$, we have $\sigma(ab) = \sigma(a) + \sigma(b)$. [This is really easy to check!] So, if $b = aq$, then $\sigma(b) = \sigma(q) + \sigma(a) \geq \sigma(a)$.

Also, if $\sigma(a) \leq \sigma(b)$, let $a = x^{\sigma(a)} a'$ and $b = x^{\sigma}(b) b'$, with $a', b' \in F[[x]]^\times$. Then, since $a' \in F[[x]]^\times$, we have $b' = qa'$. [A unit divides any element!]. So, [since $\sigma(a) \leq \sigma(b)$] we can write:

$$\begin{aligned}
b &= x^{\sigma(b)} b' \\
&= x^{\sigma(b)} q a' \\
&= x^{\sigma(b)-\sigma(a)} q x^{\sigma(a)} a' \\
&= (x^{\sigma(b)-\sigma(a)} q) a
\end{aligned}$$

So, $a \mid b$.

□

(d) Prove that $F[[x]]$ is a Euclidean domain [with size function σ].

Proof. Let $a, b \in F[[x]]$, with $a \neq 0$. [We need to show that there are $q, r \in F[[x]]$ such that $b = qa + r$, with $r = 0$ or $\sigma(r) < \sigma(a)$.] If $\sigma(b) \geq \sigma(a)$, by the previous part, we have that $a \mid b$ [i.e., this is the case when $r = 0$].

So, suppose $\sigma(b) < \sigma(a)$. Then,

$$b = 0 \cdot a + b,$$

so, since $\sigma(b) < \sigma(a)$, we have the "remainder" as b itself [i.e., $r = b$ and $q = 0$].

$\square$