Extra Credit 1
Math 456
February 19, 2007

Be careful in Problem 1(a) in the Midterm I:

1. Let $R$ be a ring and $I$ be an ideal of $R$.

(a) Prove that if $J$ is an ideal of $R$ containing $I$, then $\bar{J} \overset{\text{def}}{=} \{ \bar{a} \in R/I : a \in J \}$ is an ideal of $R/I$.

(b) Prove that if $J'$ is an ideal of $R/I$, then $J' \overset{\text{def}}{=} \{ a \in R : \bar{a} \in \bar{J}' \}$ is an ideal of $R$ containing $I$.

If, in your solution you say: “Let $\bar{a} \in \bar{J}$. Then, by definition of $\bar{J}$, we have that $a \in J.$”, you will need to justify it (or say something else)!!! It does not follow from the definition! (Why not?)

In fact, for extra credit, give an example where $I$ and $J$ are ideals of a commutative ring $R$ with unity, in which there is an $\bar{a} \in \bar{J} \overset{\text{def}}{=} \{ \bar{a} \in R/I : a \in J \}$, with $a \not\in J$.

Solution. Note first that in the case of the exam, one can get around the problem by saying something like: “Let $\bar{a} \in \bar{J}$. Then, there exists $b \in J$ such that $\bar{b} = \bar{a}$ in $\bar{J}$.” And then use, $\bar{b}$ instead of $\bar{a}$.

On the other hand, since $I \subseteq J$, we have that $a$ indeed has to be in $J$, since [as you can see in the solution of the exam], if $\bar{b} = \bar{a}$, then $a - b \in I \subseteq J$. Since $b \in J$ and $J$ is closed under addition, $a \in J$.

Now, for the extra credit: Let $R = \mathbb{Z}$, $I = (3)$ and $J = (5)$. Then, $\bar{J} = \mathbb{Z}/(3)$, since, in $\mathbb{Z}/3\mathbb{Z}$, we have $\bar{5} = \bar{2}$, $\bar{10} = \bar{1}$ and $\bar{15} = \bar{0}$. But then $\bar{2} \in \bar{J}$, but $2 \not\in J = 5\mathbb{Z}$.