Math 456 – Midterm I

Instructions: Turn in solutions for the problems that you could not do in the exam to get some partial credit. I will e-mail you, as soon as possible, the questions that you've missed. If you already know you did not do well in a question, you can start working on it right away. [Don't wait for my e-mail.]

The amount of partial credit will be decided later, depending on how the class does in the exam. [If the results are bad, I will be inclined to give more credit for these "take home" part than if the results are good.]

You cannot discuss these problems with *anyone* at all until *everyone* has turned the solutions. You can use your book and notes, though.

Deadline: Turn these in class on Wednesday (02/21).

1) Let R be a ring and I be an ideal of R.

(a) Prove that if J is an ideal of R containing I, then $\bar{J} \stackrel{\text{def}}{=} \{\bar{a} \in R/I \ : \ a \in J\}$ is an ideal of R/I.

(b) Prove that if $\bar{J}'$ is an ideal of R/I, then $J' \stackrel{\text{def}}{=} \{a \in R \ : \ \bar{a} \in \bar{J}'\}$ is an ideal of R containing I.

2) Let R be a commutative ring with identity and $a \in R$ such that $a^{n-1} \neq 0$, but $a^n = 0$, for some positive integer n. Prove that $R[x]/(ax-1) = \{\bar{0}\}$, i.e., it is the *zero ring*.

3) Let R be an integral domain, F be its field of fractions [or quotient field], and K be field such that $R \subseteq K$. Prove that there is an *injective homomorphism* $\phi : F \to K$, such that for all $a \in R$, $\phi\left(\dfrac{a}{1}\right) = a$. [**Hint:** To start, you need to find the formula for ϕ. Think of the most natural way of seeing an element of F inside of K, remembering that the image is contained in a *field*. Also, you will have to show that your formula is well defined, i.e., if $\dfrac{a}{b} = \dfrac{c}{d}$, then $\phi\left(\dfrac{a}{b}\right) = \phi\left(\dfrac{c}{d}\right)$.]

4) Prove that $\mathbb{Z}[\mathrm{i}\sqrt{3}]/(2 - \mathrm{i}\sqrt{3}) \cong \mathbb{Z}/7\mathbb{Z}$.